

EXPLICIT BIRATIONAL GEOMETRY OF THREEFOLDS OF GENERAL TYPE

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ABSTRACT. Let V be a complex nonsingular projective 3-fold of general type. We prove $P_{12}(V) > 0$ and $P_{24}(V) > 1$. We also prove that the canonical volume has an universal lower bound $\text{Vol}(V) \geq 1/2660$ and that the pluri-canonical map φ_m is birational onto its image for all $m \geq 77$. As an application of our method, we prove Fletcher's conjecture on weighted hyper-surface 3-folds with terminal quotient singularities. Another featured result is the optimal lower bound $\text{Vol}(V) \geq \frac{1}{420}$ among all those 3-folds V with $\chi(\mathcal{O}_V) \leq 1$.

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Notations

Y	a nonsingular projective variety of general type
V	a nonsingular projective 3-fold of general type
X	a minimal projective 3-fold of general type
$\text{Vol}(V), K^3$	the canonical volume
$\varphi_m = \Phi_{ mK }$	pluricanonical maps
$P_m(V), P_m(X)$	plurigenus of V, X
$\pi : X' \rightarrow X$	nonsingular birational modification
m_1	minimal positive integer with $P_{m_1} > 0$ for

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	all $m \geq m_0$
B in Sections 2, 3	the base of the induced fibration from φ_{m_0}
B, in Sections 4 ~ 7	a basket, usually a geometric basket
$\lfloor \cdot \rfloor$	round-down, taking the integral part
$\lceil \cdot \rceil$	means $-\lfloor -\cdot \rfloor$

1. Introduction

Let Y be a non-singular projective variety of dimension n . It is said to be of general type if the pluricanonical map $\varphi_m := \Phi_{|mK_Y|}$ corresponding to the linear system $|mK_Y|$ is birational into a projective space for $m \gg 0$. It is thus natural and important to ask:

Problem 1. Does there exist a constant $c(n)$, so that φ_m is birational for all $m \geq c(n)$ and for all Y with dimension n ?

When $\dim Y = 1$, it was classically known that $|mK_Y|$ gives an embedding of Y into a projective space if $m \geq 3$. When $\dim Y = 2$, Bombieri's theorem [2] says that $|mK_Y|$ gives a birational map onto the image for $m \geq 5$. This theorem has forever established the canonical classification theory for nonsingular projective surfaces of general type.

A possible approach to this problem in any dimension is to use the cohomological method via vanishing theorems. This amounts to estimating the positivity of K_Y , which is usually measured by the canonical volume

$$\text{Vol}(Y) := \limsup_{\{m \in \mathbb{Z}^+\}} \left\{ \frac{n!}{m^n} \dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y(mK_Y)) \right\}.$$

The volume is an integer when $\dim Y \leq 2$. However it's only a rational number in higher dimensions. In fact, it is almost an equivalent question to study the lower bound of the canonical volume.

Problem 2. Does there exist a constant $c'(n)$ such that $\text{Vol}(Y) \geq c'(n)$ for all varieties Y of general type with dimension n ?

Notice that a recent remarkable result of Hacon and McKernan [15], Takayama [29] and Tsuji [31] implies the affirmative answer to both Problems. However, they did not give explicit numerical bounds or the bound could only be far from realistic.

We would like to prove some explicit bounds for $c(3)$ and $c'(3)$ for 3-folds Y of general type in this paper.

Theorem 1.1 (=Theorem 8.19). *Let Y be a nonsingular projective 3-fold of general type. Then φ_m is birational for $m \geq 77$.*

Theorem 1.2 (=Theorem 7.5). *Let Y be a nonsingular projective 3-fold of general type. Then $\text{Vol}(Y) \geq \frac{1}{2660}$.*

Yet another approach for 3-folds is to study φ_{m_0} for some positive integer m_0 with φ_{m_0} non-constant. This program was first proposed by Kollár [21], and then improved by the second author.

Theorem 1.3. [9, Theorem 0.1] *Let Y be a nonsingular projective 3-fold of general type. If $P_{m_0} \geq 2$, then φ_m is birational for all $m \geq 5m_0 + 6$.*

Therefore, it is of fundamental importance to know the non-vanishing of plurigenera. In fact, Kollár and Mori proposed some related problems (see, for example, the last question of 7.74 in [22]) including the following one:

Problem 3. Does there exist a constant $c''(n)$ such that $P_m \geq 2$ (or ≥ 1) for some $m \leq c''(n)$ for all nonsingular projective varieties Y of general type with dimension n ?

We are able to answer these questions for 3-folds.

Theorem 1.4 (=Theorems 8.8, 8.9). *Let Y be a nonsingular projective 3-fold of general type. Then $P_{12} \geq 1$ and $P_{24} \geq 2$.*

An interesting application of our method is that we are able to prove Iano-Fletcher's conjecture (see [16, 15.1, 15.2]) as the following:

Theorem 1.5. *There are exactly 23 families of quasi-smooth weighted hyper-surface 3-folds X with only terminal quotient singularities and $\omega_X \cong \mathcal{O}_X(1)$.*

We now explain the idea for the proofs. The key new ingredient is, in some sense, the *classification of Reid's baskets of singularities*. Recall that for a 3-fold with canonical singularities, Reid [25] introduced the notion of baskets of singularities to compute the plurigenera. The upshot is that given a minimal 3-fold X , Reid's "virtue" baskets are uniquely determined by X . Thus to determine those baskets is a very important step.

We will introduce the notion of *packing of baskets* on certain given 3-fold, a partial ordering between baskets, which allows us to study baskets in a systematic way. In fact, there is a more refined framework which tells that for any $m \geq 3$, the set of baskets with given datum $(\chi, P_2, P_3, \dots, P_m)$ is finite.

We have discovered a key and new inequality (see inequality (5.3)):

$$2P_5 + 3P_6 + P_8 + P_{10} + P_{12} \geq \chi + 10P_2 + 4P_3 + P_7 + P_{11} + P_{13} + R,$$

with $R \geq 0$. Therefore if $P_{m_0} \geq 2$ for some $m_0 \leq 12$, then one can study φ_{m_0} and get effective results by Theorem 1.3. When $P_m \leq 1$ for all $m \leq 12$, the above inequality shows that $\chi \leq 8$. So the set of baskets with these plurigenera are finite. It is thus possible for us to classify those baskets completely, which is basically what we did.

This article is organized as the following. In Section 2, we set up some notations and generalities for the study of φ_m . In Section 3, we study Vol and P_m when $P_{m_0} \geq 2$ for some $m_0 > 0$ using the technique developed in Section 2. The main new ingredient starts from Section 4. We introduce the notion of packing in Section 4. We also describe the structure between baskets by using packings. Section 5 contains the description of baskets with given datum $(\chi, P_2, P_3, \dots, P_m)$. We remark that it gives rise to various inequalities.

Section 6 is the classification of baskets with $\chi = 1$ and $P_m \leq 1$ for $m \leq 6$. Together with the result in Section 3, we prove that $\text{Vol} \geq \frac{1}{420}$,¹ which is sharp. Section 7 presents the list of classification of baskets with $2 \leq \chi \leq 8$ and $P_m \leq 1$ for $m \leq 12$. Similarly, we get $\text{Vol} \geq \frac{1}{2660}$ for general 3-folds. With all these preparations, we prove our main theorems in Section 8, including plurigenera and the birationality for all 3-folds of general type. This is possible because for a 3-fold of general type, either $P_{m_0} \geq 2$ for some $m_0 \leq 12$, or it's classified in Sections 6 or 7. As a direct application, we prove in Section 9 a conjecture of Iano-Fletcher regarding hyper-surface 3-folds in weighted projective spaces.

There are some more applications of the techniques developed here that we will pursue in a future work.

Throughout, we work over the complex number field \mathbb{C} . We prefer to use \sim to denote the linear equivalence and \equiv means numerical equivalence.

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2. Technical preparations

Definition 2.1. Let L be a divisor on a nonsingular projective variety Y with $n_L := h^0(Y, \mathcal{O}_Y(L)) - 1 \geq 1$. Pick a basis $s_0, \dots, s_{n_L} \in H^0(Y, \mathcal{O}_Y(L))$. For any point $x \in Y$, we define a rational map $\Phi_{|L|} : Y \dashrightarrow \mathbb{P}^{n_L}$ by sending x to $[s_0(x), \dots, s_{n_L}(x)]$. $\Phi_{|L|}$ is usually said to be *the rational map associated to $|L|$* .

First of all we list the birationality principles which we will frequently use in our arguments:

¹The authors were informed by Lei Zhu that she obtained the same lower bound independently.

2.2. Birationality principles. Let Y be a nonsingular projective variety on which there are two divisors D and M . Suppose $|M|$ is base point free. Take the Stein factorization of $\Phi_{|M|}$:

$$Y \xrightarrow{f} W \longrightarrow \mathbb{P}^{h^0(Y, M)-1}$$

where f is a fibration onto a normal variety W . Then the rational map $\Phi_{|D+M|}$ is birational onto its image if one of the following conditions is satisfied:

- (i) ([30, Lemma 2]) $\dim \Phi_{|M|}(Y) \geq 2$, $|D| \neq \emptyset$ and $\Phi_{|D+M||_S}$ is birational for a general member S of $|M|$.
- (ii) ([7, §2.1]) $\dim \Phi_{|M|}(Y) = 1$, $\Phi_{|D+M|}$ can separate different general fibers of f and $\Phi_{|D+M||_F}$ is birational for a general fiber F of f .

Remark 2.3. For the condition 2.2 (ii), one knows that $\Phi_{|D+M|}$ can separate different general fibers of f whenever $\dim \Phi_{|M|}(Y) = 1$, W is a rational curve and D is an effective divisor. In fact, since $|M|$ can separate different fibers of f , so can $|D+M|$. We do not care too much about the situation when W is an irrational curve, since the results and technique in [4] are sufficient for our purpose here.

2.4. Invariants of the fibration. Let V be a smooth projective 3-fold and $f : V \rightarrow B$ a fibration onto a nonsingular curve B . There is a spectral sequence,

$$E_2^{p,q} := H^p(B, R^q f_* \omega_V) \implies E^n := H^n(V, \omega_V).$$

By Serre duality and [21, Corollary 3.2, Proposition 7.6], one has the torsion-freeness of the sheaves $R^i f_* \omega_V$ and the following formulae:

$$\begin{aligned} h^2(\mathcal{O}_V) &= h^1(B, f_* \omega_V) + h^0(B, R^1 f_* \omega_V), \\ q(V) &:= h^1(\mathcal{O}_V) = g(B) + h^1(B, R^1 f_* \omega_V). \end{aligned}$$

2.5. Reduction step. Let V be a nonsingular projective 3-fold of general type. By the 3-dimensional MMP (see for instance [20, 22, 26]), we can pick a minimal model X of V and allow X to have at worst \mathbb{Q} -factorial terminal singularities. Denote by K_X a canonical divisor of X . We recall the following birational invariants.

$$\begin{aligned} P_m(V) &:= h^0(V, \mathcal{O}_V(mK_V)) = h^0(X, \mathcal{O}_X(mK_X)) =: P_m(X); \\ \chi(\mathcal{O}_V) &= \chi(\mathcal{O}_X); \\ \text{Vol}(V) &:= \limsup \frac{3!}{m^3} h^0(V, mK_V) = K_X^3. \end{aligned}$$

Note that the rational maps $\Phi_{|mK_V|}$ and $\varphi_m := \Phi_{|mK_X|}$ are birationally equivalent. Sometimes we simply denote by K^3 the canonical volume of V and X .

From this point of view, it suffices to prove our main theorem only for minimal 3-folds X .

(♡) Throughout, X will be an arbitrary minimal 3-fold of general type with at worst \mathbb{Q} -factorial terminal singularities. The integer m_0 always denotes a positive (most likely, minimal) integer with

$$P_{m_0} = P_{m_0}(X) := \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(m_0 K_X)) \geq 2$$

where K_X is a canonical divisor of X . By minimal, we mean that K_X is nef. Define $r(X)$ to be the minimal positive integer such that $r(X)K_X$ is Cartier. It is already known that $r(X)$ is uniquely determined by the birational equivalence class of X . So it is a birational invariant within the category of 3-folds having at worst canonical singularities.

2.6. Set up for φ_{m_0} . We study the m_0 -canonical map of X :

$$\varphi_{m_0} : X \dashrightarrow \mathbb{P}^{P_{m_0}-1}$$

which is only a rational map. First of all we fix an effective Weil divisor $K_{m_0} \sim m_0 K_X$. By Hironaka's big theorem, we can take successive blow-ups $\pi : X' \rightarrow X$ such that:

- (i) X' is smooth;
- (ii) the movable part of $|m_0 K_{X'}|$ is base point free;
- (iii) the support of the union of $\pi^*(K_{m_0})$ and the exceptional divisors is of simple normal crossings.

Set $g_{m_0} := \varphi_{m_0} \circ \pi$. Then g_{m_0} is a morphism by assumption. Let $X' \xrightarrow{f} B \xrightarrow{s} W'$ be the Stein factorization of g_{m_0} with W' the image of X' through g_{m_0} . In summary, we have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f} & B \\ \pi \downarrow & \searrow g_{m_0} & \downarrow s \\ X & \xrightarrow{\varphi_{m_0}} & W' \end{array}$$

Let us recall the definition of $\pi^*(K_X)$. We can write $r(X)K_{X'} = \pi^*(r(X)K_X) + E_\pi$ where E_π is a sum of exceptional divisors. We define

$$\pi^*(K_X) := K_{X'} - \frac{1}{r(X)} E_\pi.$$

So, whenever we take the round-up of $m\pi^*(K_X)$, we always have

$$\lceil m\pi^*(K_X) \rceil \leq mK_{X'}$$

for any integer $m > 0$. Denote by M_{m_0} the movable part of $|m_0 K_{X'}|$. One has

$$m_0 \pi^*(K_X) = M_{m_0} + E'_{m_0}$$

for an effective \mathbb{Q} -divisor E'_{m_0} because

$$h^0(X', \lceil m_0 \pi^*(K_X) \rceil) = h^0(X', \lceil m_0 \pi^*(K_X) \rceil) = P_{m_0}(X') = P_{m_0}(X).$$

On the other hand, one can write $m_0 K_{X'} = m_0 \pi^*(K_X) + E_{m_0}$ where E_{m_0} is an effective \mathbb{Q} -divisor as a \mathbb{Q} -sum of distinct exceptional divisors. Thus $m_0 K_{X'} = M_{m_0} + Z_{m_0}$ where $Z_{m_0} := E'_{m_0} + E_{m_0}$ is exactly the fixed part of $|m_0 K_{X'}|$.

If $\dim(B) \geq 2$, a general member S of $|M_{m_0}|$ is a nonsingular projective surface of general type by Bertini's theorem and by the easy addition formula for Kodaira dimension.

If $\dim(B) = 1$, a general fiber S of f is an irreducible smooth projective surface of general type, still by the easy addition formula for Kodaira dimension. We may write

$$M_{m_0} = \sum_{i=1}^{a_{m_0}} S_i \equiv a_{m_0} S$$

where the S_i is a smooth fiber of f for all i and $a_{m_0} \geq P_{m_0}(X) - 1$ by considering the degree of a non-degenerate curve in a projective space.

Define the positive integer

$$p = \begin{cases} 1 & \text{if } \dim(B) \geq 2 \\ a_{m_0} & \text{if } \dim(B) = 1. \end{cases}$$

Definition 2.7. In both cases regarding to $\dim(B)$, we call S a *generic irreducible element* of the linear system $|M_{m_0}|$. Denote by $\sigma : S \rightarrow S_0$ the blow-down onto the smooth minimal model S_0 .

By abuse of concepts we define a *generic irreducible element of any given movable linear system* on any projective variety in a similar way.

2.8. Type of f . To simplify our statements, we say that the fibration f induced from φ_{m_0} is of type III (resp. II, I) if $\dim B = 3$ (resp. 2, 1). In the case of type I, we distinguish into subcases as the following:

$$\begin{cases} I_q & \text{if } g(B) > 0, \\ I_3 & \text{if } P_{m_0} \geq 3, \\ I_p & \text{if } g(B) = 0, \ p_g(S) > 0, \\ I_n & \text{if } g(B) = 0, \ p_g(S) = 0. \end{cases}$$

2.9. Assumptions. Keep the same setup as in 2.6. Let m be a positive integer. We need some assumptions to estimate K^3 and to study φ_m .

- (1) Take a generic irreducible element S of $|M_{m_0}|$. Assume that there is a base point free complete linear system $|G|$ on S . Denote by C a generic irreducible element of $|G|$.
- (2) Assume there is a rational number $\beta > 0$ such that $\pi^*(K_X)|_S - \beta C$ is numerically equivalent to an effective \mathbb{Q} -divisor on S .
- (3) The linear system $|mK_{X'}|$ separates different generic irreducible elements of $|M_{m_0}|$ (namely, $\Phi_{|mK_{X'}|}(S') \neq \Phi_{|mK_{X'}|}(S'')$ for two different irreducible elements S', S'' of $|M_{m_0}|$).

- (4) The linear system $|mK_{X'}|_{|S}$ on S (as a sub-linear system of $|mK_{X'}|_{|S}|$) separates different generic irreducible elements of $|G|$. Or sufficiently, the complete linear system

$$|K_S + \lceil (m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0} \rceil_{|S}|$$

separates different generic irreducible elements of $|G|$.

Set the following quantities:

$$\begin{aligned}\xi &:= (\pi^*(K_X) \cdot C)_{X'}; \\ \alpha &:= (m-1 - \frac{m_0}{p} - \frac{1}{\beta})\xi; \\ \alpha_0 &:= \lceil \alpha \rceil.\end{aligned}$$

Under Assumptions 2.9 (1), (2), clearly one has

$$K^3 \geq \frac{p}{m_0}\pi^*(K_X)^2 \cdot S \geq \frac{p\beta}{m_0}(\pi^*(K_X) \cdot C) = \frac{p\beta}{m_0}\xi. \quad (2.1)$$

So it suffices to estimate the rational number $\xi := (\pi^*(K_X) \cdot C)_{X'}$ in order to obtain the lower bound of K^3 .

2.10. Suppose that $P_{m_0} \geq 2$ and Assumption 2.9(1), 2.9(2) hold. Let m be an integer such that $m > 1 + \frac{m_0}{p} + \frac{1}{\beta}$. Let

$$\mathcal{L}_m := (m-1)\pi^*(K_X) - \frac{1}{p}E'_{m_0}$$

be a \mathbb{Q} -divisor on X' . Clearly, we have

$$|K_{X'} + \lceil \mathcal{L}_m \rceil| \subset |mK_{X'}|.$$

Noting that $\mathcal{L}_m - S \equiv (m-1 - \frac{m_0}{p})\pi^*(K_X)$ is nef and big, then the Kawamata-Viehweg vanishing theorem ([19, 32]) yields the surjective map

$$H^0(X', K_{X'} + \lceil \mathcal{L}_m \rceil) \rightarrow H^0(S, (K_{X'} + \lceil \mathcal{L}_m \rceil)_{|S}). \quad (2.2)$$

Since S is a generic irreducible element of a free linear system, one has $\lceil * \rceil_{|S} \geq \lceil * \rceil_S$ for any divisor $*$. It follows that

$$(K_{X'} + \lceil \mathcal{L}_m \rceil)_{|S} \geq K_{X'|S} + \lceil \mathcal{L}_m \rceil_S = K_S + \lceil (\mathcal{L}_m - S) \rceil_S. \quad (2.3)$$

By Assumption 2.9(2), there is an effective \mathbb{Q} -divisor H on S such that $\frac{1}{\beta}\pi^*(K_X)_{|S} \equiv C + H$. We now consider

$$\mathcal{D}_m := (\mathcal{L}_m - S)_{|S} - H$$

on S . The divisor $\mathcal{D}_m - C \equiv (m-1 - \frac{m_0}{p} - \frac{1}{\beta})\pi^*(K_X)_{|S}$ is nef and big. Thus the Kawamata-Viehweg vanishing theorem again gives the following surjective map

$$H^0(S, K_S + \lceil \mathcal{D}_m \rceil) \longrightarrow H^0(C, K_C + D), \quad (2.4)$$

where $D := [\mathcal{D}_m - C]|_C$ is a divisor on C . Because C is a generic irreducible element of a free linear system, we have $D \geq [(\mathcal{D}_m - C)|_C]$ similarly. A simple calculation gives

$$\deg(D) \geq (\mathcal{D}_m - C) \cdot C = (m - 1 - \frac{m_0}{p} - \frac{1}{\beta})\xi = \alpha \quad (2.5)$$

Proposition 2.11. *Keep the notation as above. Then $P_m(X) \geq 2$ for all integer $m > 1 + \frac{m_0}{p} + \frac{1}{\beta}$.*

Proof. This is clear from the inclusion $|K_{X'} + [\mathcal{L}_m]| \subset |mK_{X'}|$, surjections (2.2) and (2.4), together with (2.3), (2.5) and the fact that $g(C) \geq 2$. \square

Theorem 2.12. *Let $m > 0$ be an integer satisfying Assumptions 2.9(1), 2.9(2). The inequality*

$$\xi \geq \frac{\deg(K_C) + \alpha_0}{m}$$

holds if one of the following conditions is satisfied:

- (i) $\alpha > 1$;
- (ii) $\alpha > 0$ and C is an even divisor on S .

Furthermore if Assumptions 2.9 (1) through (4) are satisfied. The map φ_m is birational onto its image when one of the following conditions is satisfied:

- (i) $\alpha > 2$;
- (ii) $\alpha \geq 2$ and C is not a hyper-elliptic curve on S .

Remark 2.13. In particular the inequality $\xi \geq \frac{\deg(K_C) + \alpha_0}{m}$ in Theorem 2.12 implies $\xi \geq \frac{\deg(K_C)}{1 + \frac{m_0}{p} + \frac{1}{\beta}}$ since, whenever m is big enough so that $\alpha > 1$,

$$m\xi \geq \deg(K_C) + \alpha_0 \geq \deg(K_C) + \alpha = \deg(K_C) + (m - 1 - \frac{m_0}{p} - \frac{1}{\beta})\xi.$$

Proof of Theorem 2.12. We keep the notation as above. Denote by $|M_m|$ the movable part of $|mK_{X'}|$ and by $|M'_m|$ the movable part of $|K_{X'} + [\mathcal{L}_m]|$. Clearly, one has $M_m \geq M'_m$ by definition.

Let $|N_m|$ be the movable part of $|(K_{X'} + [\mathcal{L}_m])|_S|$. Applying Lemma 2.7 of [7] to the surjective map (2.2), we have

$$M'_m|_S \geq N_m.$$

We now claim that $|K_C + D|$ is base point free. To see this, if $\alpha > 1$, then $\deg(D) \geq \alpha_0 \geq 2$. Thus $|K_C + D|$ is free. If C is an even divisor, then $\deg(D)$ is even since $[\mathcal{D}_m - C]$ is an integral divisor on S . If moreover $\alpha > 0$, then $\deg(D) \geq 2$. Therefore $|K_C + D|$ is base point free. Hence in both cases, the movable part of $|K_C + D|$ is itself.

Let $|N'_m|$ be the movable part of $|K_S + \lceil \mathcal{D}_m \rceil|$. Applying Lemma 2.7 of [7] to surjective map (2.4), we have

$$N'_m|_C \geq K_C + D.$$

Note that $N_m \geq N'_m$. Combining all these gives

$$m\pi^*K_X \cdot C \geq (N'_m \cdot C)_S \geq 2g(C) - 2 + \deg(D).$$

Therefore,

$$\xi \geq \frac{\deg(K_C) + \alpha_0}{m}.$$

Next we prove the birationality. Assumption 2.9.(3) says that $|mK_{X'}|$ can separate different irreducible elements of $|M_{m_0}|$. The birationality principle 2.2 permits us only to verify the birationality of $|mK_{X'}|_S$ on a generic irreducible element S of $|M_{m_0}|$.

By Assumptions 2.9.(1) and 2.9.(3), there is a base point free linear system $|G|$ on S and the linear system $|mK_{X'}|_S$ on S separates different generic irreducible elements of $|G|$. The birationality principle 2.2 reduces the problem to verify the birationality of $(|mK_{X'}|_S)_C$ on a generic irreducible element C of $|G|$. In fact we will prove this for a sub-linear system of $(|mK_{X'}|_S)_C$.

From the above discussion, we only need to verify that $|K_C + D|$ gives a birational map onto the image of C . This is the case whenever either $\deg(D) \geq 3$ or C is non-hyperelliptic and $\deg(D) \geq 2$. This completes the proof. \square

The following lemma has already appeared in a couple of unpublished preprints of the second author. In order to make this paper more self-contained we would like to collect the proof here. In fact, the special case that $p_g \geq 2$ has been published in [11, Lemma 3.7].

Lemma 2.14. *Keep the same notation as in 2.6, 2.9 and Theorem 2.12. Assume $B = \mathbb{P}^1$. Let $f : X' \rightarrow \mathbb{P}^1$ be an induced fibration of φ_{m_0} . Then one can find a sequence of rational numbers $\{\beta_n\}$ with $\lim_{n \rightarrow +\infty} \beta_n = \frac{p}{m_0+p}$ such that $\pi^*(K_X)|_S - \beta_n \sigma^*(K_{S_0})$ is \mathbb{Q} -linearly equivalent to an effective \mathbb{Q} -divisor H_n .*

Proof. We use Kollár's technique in [21]. One has $\mathcal{O}_B(p) \hookrightarrow f_*\omega_{X'}^{m_0}$. The inclusion relation between divisors gives the inclusion of sheaves:

$$f_*\omega_{X'/B}^{t_0p} \hookrightarrow f_*\omega_{X'}^{t_0p+2t_0m_0}$$

for any big integer t_0 .

For any positive integer k , we know in 2.6 that M_k denotes the movable part of $|kK_{X'}|$. Note that $f_*\omega_{X'/B}^{t_0p}$ is generated by global sections since it is semi-positive according to Viehweg ([33]). So any local section of $f_*\omega_{X'/B}^{t_0p}$ can be extended to be a global one. On the other hand, $|t_0p\sigma^*(K_{S_0})|$ is base point free and is exactly the movable

part of $|t_0pK_S|$ by Bombieri [2] or Reider [28]. Clearly one has the following relation:

$$(t_0p + 2t_0m_0)\pi^*(K_X)|_S \geq M_{t_0p+2t_0m_0}|_S \geq t_0p\sigma^*(K_{S_0}).$$

Set $a_0 := t_0p + 2t_0m_0$ and $b_0 := t_0p$. Then there is an effective \mathbb{Q} -divisor I'_0 on S such that

$$a_0\pi^*(K_X)|_S =_{\mathbb{Q}} b_0\sigma^*(K_{S_0}) + I'_0.$$

Thus $\pi^*(K_X)|_S =_{\mathbb{Q}} \frac{b_0}{a_0}\sigma^*(K_{S_0}) + I_0$ with $I_0 = \frac{1}{a_0}I'_0$ still an effective \mathbb{Q} -divisor.

Case 1. First we consider the case $p \geq 2$.

We use an induction on the basis of the numbers a_0 and b_0 . Suppose that we have defined a_l and b_l such that the following is satisfied with $l = n$:

$$a_n\pi^*(K_X)|_S \geq b_n\sigma^*(K_{S_0}).$$

We will define a_{l+1} and b_{l+1} inductively such that the above inequality is satisfied with $l = n + 1$. By assumption we know that $a_n\pi^*(K_X)$ supports on a divisor with normal crossings. Then the Kawamata-Viehweg vanishing theorem implies the surjective map:

$$H^0(K_{X'} + \lceil a_n\pi^*(K_X) \rceil + S) \longrightarrow H^0(S, K_S + \lceil a_n\pi^*(K_X) \rceil|_S).$$

One sees the following relations:

$$\begin{aligned} |K_{X'} + \lceil a_n\pi^*(K_X) \rceil + S|_S &= |K_S + \lceil a_n\pi^*(K_X) \rceil|_S \\ &\supset |K_S + b_n\sigma^*(K_{S_0})| \\ &\supset |(b_n + 1)\sigma^*(K_{S_0})|. \end{aligned}$$

Denote by M'_{a_n+1} the movable part of $|(a_n + 1)K_{X'} + S|$. Applying Lemma 2.7 of [7], one gets $M'_{a_n+1}|_S \geq (b_n + 1)\sigma^*(K_{S_0})$. Re-modifying our original π , if necessary, such that $|M'_{a_n+1}|$ is base point free. In particular, M'_{a_n+1} is nef. Since X is of general type $|mK_X|$ gives a birational map whenever m is big enough. Thus we see that M'_{a_n+1} is big as we fix a very big t_0 in advance.

Now the Kawamata-Viehweg vanishing theorem again gives

$$\begin{aligned} |K_{X'} + M'_{a_n+1} + S|_S &= |K_S + M'_{a_n+1}|_S \\ &\supset |K_S + (b_n + 1)\sigma^*(K_{S_0})| \\ &\supset |(b_n + 2)\sigma^*(K_{S_0})|. \end{aligned}$$

We may repeat the above procedure. Denote by M'_{a_n+t} the movable part of $|K_{X'} + M'_{a_n+t-1} + S|$ for $t \geq 2$. For the same reason, we may assume $|M'_{a_n+t}|$ to be base point free. Inductively one has:

$$M'_{a_n+2}|_S \geq (b_n + 2)\sigma^*(K_{S_0})$$

and in general

$$M'_{a_n+t}|_S \geq (b_n + t)\sigma^*(K_{S_0})$$

Just take $t = p$ and set $a_{n+1} := a_n + p + m_0$ and $b_{n+1} = b_n + p$. Noting that

$$|K_{X'} + M'_{a_n+p-1} + S| \subset |(a_n + p + m_0)K_{X'}|$$

and applying Lemma 2.7 of [7] again, one has

$$a_{n+1}\pi^*(K_X)|_S \geq M_{a_n+p+m_0}|_S \geq M'_{a_n+p}|_S \geq b_{n+1}\sigma^*(K_{S_0}).$$

Set $\beta_n := \frac{b_n}{a_n}$. Clearly $\lim_{n \rightarrow +\infty} \beta_n = \frac{p}{m_0+p}$. We have proved the lemma when $p \geq 2$.

Case 2. The lemma at the case $p = 1$ can be proved similarly with a simpler induction. We omit the proof and leave it to readers as an exercise. \square

3. Lower bound of the volume and non-vanishing

In this section, we are going to utilize the general method developed in Section 2.

Theorem 3.1. *Let V be a nonsingular projective 3-fold of general type with $P_{m_0} \geq 2$. Then*

- (i) $\text{Vol}(V) \geq \frac{10}{m_0^2(3m_0+2)}$ for type III.
- (ii) $\text{Vol}(V) \geq \frac{4}{m_0^2(3m_0+2)}$ for type II.
- (iii) $\text{Vol}(V) \geq \frac{36}{5m_0(m_0+2)^2}$ for type I_3 .
- (iv) $\text{Vol}(V) \geq \frac{11}{12m_0(m_0+1)^2}$ in general.

Proof. Take a minimal model X of V . We study $|mK_X|$ on X . Keep the same setup as in 2.6. Then $\text{Vol}(V) = K_X^3$ as we have known.

Part (i). For type III, i.e. $\dim(B) = 3$, we know that $p = 1$ by definition. In this case we pick $S \sim M_{m_0}$ and that $|S|$ gives a generically finite morphism. Set $G := S|_S$. Then $|G|$ is base point free and $\varphi_{|G|}$ gives a generically finite map. So a generic irreducible element C of $|G|$ is a smooth curve.

If $\varphi_{|G|}$ gives a birational map, then $\dim \varphi_{|G|}(C) = 1$ for a general member C . The Riemann-Roch and Clifford's theorem on C says $C^2 = G \cdot C \geq 2$. If $\varphi_{|G|}$ gives a generically finite map of degree ≥ 2 , since $h^0(S, G) \geq h^0(X', S) - 1 \geq 3$, [6, Lemma 2.2] gives $C^2 \geq 2h^0(S, G) - 4 \geq 2$. Anyway we have $C^2 \geq 2$. So $\deg(K_C) = (K_S + C) \cdot C > 2C^2 \geq 4$. We see $\deg(K_C) \geq 6$ because it is even.

One may take $\beta = \frac{1}{m_0}$ since $m_0\pi^*(K_X)|_S \geq C$. Now if we take $m \gg 0$ such that $\alpha > 1$ then Theorem 2.12 gives:

$$m\xi \geq \deg(K_C) + (m - 1 - m_0 - \frac{1}{\beta})\xi.$$

This gives $\xi \geq \frac{6}{2m_0+1}$. Take $m = 3m_0 + 2$. Then $\alpha = (m - 2m_0 - 1)\xi > 3$. So by Theorem 2.12 again, $\xi \geq \frac{10}{3m_0+2}$. It follows that $K^3 \geq \frac{10}{(3m_0+2)m_0^2}$.

Part (ii). If $\dim(B) = 2$, we pick $S \sim M_{m_0}$ and $|G| := |S|_S$ is composed with a pencil of curves.

A generic irreducible element C of $|G|$ is a smooth curve of genus ≥ 2 , so $\deg(K_C) \geq 2$. Furthermore we have $h^0(S, G) \geq h^0(X', S) - 1 \geq 2$. So $G \equiv \tilde{a}C$ for $\tilde{a} \geq 1$. This means $m_0\pi^*(K_X)|_S \geq S|_S \geq_{\text{num}} C$. So we may take $\beta = \frac{1}{m_0}$.

Now take a $m \gg 0$. Remark 2.13 gives $\xi \geq \frac{2}{2m_0+1}$. Take $m = 3m_0+2$. Then $\alpha > 1$. One gets $\xi \geq \frac{4}{3m_0+2}$ by Theorem 2.12. So $K^3 \geq \frac{4}{(3m_0+2)m_0^2}$.

Part (iii). Take $S \sim M_{m_0}$ and $G := 4\sigma^*(K_{S_0})$, where S_0 is the minimal model of S . The surface theory tells us that $|G|$ is base point free and a generic irreducible element C of $|G|$ is a smooth curve. Because

$$\deg(K_C) = (K_S + C) \cdot C \geq (\pi^*(K_X)|_S + C) \cdot C > C^2 \geq 16,$$

again we see $\deg(K_C) \geq 18$.

We know that $\pi^*(K_X)|_S - \tilde{\beta}_n\sigma^*(K_{S_0})$ is numerically equivalent to an effective \mathbb{Q} -divisor for a rational number sequence $\{\tilde{\beta}_n\}$ with $\tilde{\beta}_n \mapsto \frac{p}{p+m_0} \geq \frac{2}{m_0+2}$. Take $\beta_n := \frac{1}{4}\tilde{\beta}_n$. Then $\pi^*(K_X)|_S - \beta_n C$ is numerically equivalent to an effective \mathbb{Q} -divisor. We can take a rational number $\beta = \frac{1}{2(m_0+2)} - \delta$ with $0 < \delta \ll 1$.

If we take $m \gg 0$, then $\xi \geq \frac{18}{1+\frac{m_0}{2}+\frac{1}{\beta}}$ by Remark 2.13. So $\xi \geq \frac{36}{5(m_0+2)}$ by letting δ go to 0. One gets $K^3 \geq \frac{36}{5m_0(m_0+2)^2} \geq \frac{4}{(3m_0+2)m_0^2}$.

Part (iv). It remains to study the case $\dim(B) = 1$. When $q > 0$, one has $K^3 \geq \frac{1}{22}$ by [5] and one can easily verify the inequality $K^3 \geq \frac{11}{12m_0(m_0+1)^2}$. So we may assume $q = 0$. So B is a rational curve. We set $G := 4\sigma^*(K_{S_0})$. Again we see $\deg(K_C) \geq 18$. We know that $\pi^*(K_X)|_S - \tilde{\beta}_n\sigma^*(K_{S_0})$ is numerically equivalent to an effective \mathbb{Q} -divisor for a rational number sequence $\{\tilde{\beta}_n\}$ with $\tilde{\beta}_n \mapsto \frac{p}{p+m_0} \geq \frac{1}{m_0+1}$. Take $\beta_n := \frac{1}{4}\tilde{\beta}_n$. Then $\pi^*(K_X)|_S - \beta_n C$ is numerically equivalent to an effective \mathbb{Q} -divisor. We know $\beta_n \mapsto \frac{1}{4m_0+4}$ whenever $p = 1$. We thus take $\beta = \frac{1}{4m_0+4} - \delta$ for some $0 < \delta \ll 1$.

When $m \gg 0$, Remark 2.13 gives $\xi \geq \frac{18}{5m_0+5}$. Take $m = 6m_0 + 6$. Then $\alpha > 3$ and Theorem 2.12 gives $\xi \geq \frac{11}{3m_0+3}$. So $K^3 \geq \frac{11}{12m_0(m_0+1)^2}$. \square

3.2. Refinement of lower bounds of K^3 . Indeed, for small m_0 , we can improve the lower bound of the canonical volume. We study some special cases that occur in our paper.

For example, in type III, assume $m_0 = 11$. Then $\xi \geq \frac{6}{23}$ by taking $m \gg 0$. Next take $m = 27$. By Theorem 2.12, we have $\xi \geq \frac{8}{27}$. So inequality (2.1) gives $K^3 \geq \frac{8}{3267}$.

Let's assume $m_0 = 8$ for type II as another example. Then $\beta \geq \frac{1}{8}$. One has already $\xi \geq \frac{2}{17}$ by taking $m \gg 0$. Take $m = 26$. Then $\alpha \geq \frac{18}{17} > 1$. We get $\xi \geq \frac{2}{13}$. Take $m = 24$. Then $\alpha > 1$. One gets $\xi \geq \frac{1}{6}$. So inequality (2.1) gives $K^3 \geq \frac{1}{384}$.

A patient reader should have no difficulty to check the following table on the lower bound of K^3 for small m_0 . We tag it as:

Table A

m_0	2	3	4	5	6	7	8	9	10	11	12
III	$\frac{1}{7}$	$\frac{8}{81}$	$\frac{1}{22}$	$\frac{8}{325}$	$\frac{1}{72}$	$\frac{4}{441}$	$\frac{1}{160}$	$\frac{4}{891}$	$\frac{2}{625}$	$\frac{8}{3267}$	$\frac{1}{527}$
II	$\frac{1}{8}$	$\frac{2}{45}$	$\frac{1}{52}$	$\frac{1}{100}$	$\frac{1}{162}$	$\frac{1}{1029}$	$\frac{1}{384}$	$\frac{1}{1053}$	$\frac{1}{725}$	$\frac{1}{968}$	$\frac{1}{1224}$
I_3	$\frac{1}{8}$	$\frac{2}{45}$	$\frac{1}{52}$	$\frac{1}{100}$	$\frac{1}{162}$	$\frac{1}{1029}$	$\frac{1}{384}$	$\frac{1}{1053}$	$\frac{1}{725}$	$\frac{1}{968}$	$\frac{1}{1224}$
general	$\frac{5}{96}$	$\frac{5}{264}$	$\frac{1}{108}$	$\frac{1}{192}$	$\frac{5}{1554}$	$\frac{5}{2408}$	$\frac{5}{3456}$	$\frac{1}{954}$	$\frac{1}{1276}$	$\frac{5}{8448}$	$\frac{5}{10764}$

We now study the non-vanishing problem of plurigenera.

Definition 3.3. Let X be a minimal projective 3-fold of general type. Define $m_1 := m_1(X)$ to be the smallest positive integer such that $P_m(X) > 0$ for all $m \geq m_1(X)$. Clearly $m_1(X)$ is a birational invariant of X . One knows $m_1(X) < +\infty$ by Matsusaka's big theorem.

In fact, by Proposition 2.11, one has $m_1 \leq 5m_0 + 6$ already. We will need the following easy lemma to get a better bound.

Lemma 3.4. *Let S be a nonsingular projective surface of general type. Denote by $\sigma : S \rightarrow S_0$ the blow-down onto its minimal model S_0 . Let Q be a \mathbb{Q} -divisor on S . Then $h^0(S, K_S + \lceil Q \rceil) \geq 2$ under one of the following conditions:*

- (i) $p_g(S) > 0$, $Q \equiv \sigma^*(K_{S_0}) + Q_1$ for some nef and big \mathbb{Q} -divisor Q_1 on S ;
- (ii) $p_g(S) = 0$, $Q \equiv 2\sigma^*(K_{S_0}) + Q_2$ for some nef and big \mathbb{Q} -divisor Q_2 on S ;

Proof. First of all $h^0(S, 2K_S) = h^0(S, 2K_{S_0}) > 0$ by the Riemann-Roch theorem on S , which is a surface of general type. Fix an effective divisor $R_0 \sim l\sigma^*(K_{S_0})$, where $l = 1, 2$ in cases (i) and (ii) respectively. Then R_0 is nef and big and R_0 is 1-connected by [23, Lemma 2.6]. The Kawamata-Viehweg vanishing theorem says $H^1(S, K_S + \lceil Q \rceil - R_0) = 0$ which gives the surjective map:

$$H^0(S, K_S + \lceil Q \rceil) \longrightarrow H^0(R_0, K_{R_0} + G_{R_0})$$

where $G_{R_0} := (\lceil Q \rceil - R_0)|_{R_0}$ with $\deg(G_{R_0}) \geq (Q - R_0)R_0 = Ql \cdot R_0 > 0$. The 1-connectedness of R_0 allows us to utilize the Riemann-Roch (see Chapter II, [1]) as in the usual way. Note that S is of general type. So $K_{S_0}^2 > 0$ and $\deg(K_{R_0}) = 2p_a(R_0) - 2 = (K_S + R_0)R_0 \geq 2$. By the Riemann-Roch theorem on the 1-connected curve R_0 , we have

$$h^0(R_0, K_{R_0} + G_{R_0}) \geq \deg(K_{R_0} + G_{R_0}) + 1 - p_a(R_0) \geq p_a(R_0) \geq 2.$$

Hence $h^0(S, K_S + \lceil Q \rceil) \geq 2$. \square

Proposition 3.5. *Let X be a minimal projective 3-fold of general type with $P_{m_0} \geq 2$. Keep the same notation as in 2.6. Then m_1 has an upper bound under each of the following situations:*

- (i) $P_m \geq 2$ for all $m \geq 2m_0$ for type III;
- (ii) $P_m \geq 2$ for all $m \geq 2m_0$ for type II;
- (iii) $P_m \geq 2$ for all $m \geq 2m_0 + 3$ for type I_p ;
- (iv) $P_m \geq 2$ for all $m \geq 3m_0 + 4$ for type I_n ;
- (v) $P_m \geq 2$ for all $m \geq \lfloor \frac{3m_0}{2} \rfloor + 4$ for type I_3 .

Proof. We keep the notation as in Section 2.

(i). For type III, one can pick $\beta = \frac{1}{m_0}$, thus by Proposition 2.11, we have $P_m \geq 2$ for all $m > 2m_0 + 1$. Now if $m = 2m_0 + 1$, the surjection (2.2) and (2.3) lead us to consider non-vanishing of $H^0(S, K_S + \lceil m_0 \pi^* K_X|_S \rceil)$. Let L be an element in $|M_{m_0}|_S$, then clearly, $h^0(S, K_S + L) = \chi(S, K_S + L) \geq 2$ by Riemann-Roch theorem. Hence $P_{2m_0+1} \geq 2$. Also, $P_{2m_0} \geq P_{m_0} \geq 2$. Therefore, we have $m_1 \leq 2m_0$.

(ii). For type II, since we can take $\beta = \frac{1}{m_0}$, exactly the same proof shows that $m_1 \leq 2m_0$.

(iii). For type I, Lemma 2.14 gives that there is a sequence of rational numbers $\{\beta_n\}$ with $\beta_n \mapsto \frac{p}{m_0+p} \geq \frac{1}{m_0+1}$ such that

$$\pi^*(K_X)|_S - \beta_n \sigma^*(K_{S_0}) \equiv H_n$$

for an effective \mathbb{Q} -divisor H_n .

We consider

$$\mathcal{D}'_m := (\mathcal{L}_m - S)|_S - (m - 1 - \frac{m_0}{p})H_n \equiv (m - 1 - \frac{m_0}{p})\beta_n \sigma^*(K_{S_0}).$$

If $h^0(S, K_S + \lceil \mathcal{D}'_m \rceil) \geq 2$, then so is $h^0(S, K_S + \lceil (\mathcal{L}_m - S)|_S \rceil)$. It follows that $P_m \geq 2$ by (2.2) and the surjection (2.3).

In case I_p , we can pick $\beta_n = \frac{1}{m_0+1} - \delta$ for some $0 < \delta \ll 1$. So when $m \geq 2m_0 + 3$, $(m - 1 - \frac{m_0}{p})\beta_n > 1$. By Lemma 3.4, we have $h^0(S, K_S + \lceil \mathcal{D}'_m \rceil) \geq 2$. Thus $m_1 \leq 2m_0 + 3$.

In case I_n , similarly, we can pick $\beta_n = \frac{1}{m_0+1} - \delta$ for some $0 < \delta \ll 1$. So when $m \geq 3m_0 + 4$, $(m - 1 - \frac{m_0}{p})\beta_n > 2$. By Lemma 3.4, we have $h^0(S, K_S + \lceil \mathcal{D}'_m \rceil) \geq 2$. Thus $m_1 \leq 3m_0 + 4$.

In case I_3 , we can pick $\beta_n = \frac{2}{m_0+2} - \delta$ for some $0 < \delta \ll 1$. So when $m \geq \lfloor \frac{3m_0}{2} \rfloor + 4$, we have $(m - 1 - \frac{m_0}{p})\beta_n > 2$ and $P_m \geq 2$. Thus $m_1 \leq \lfloor \frac{3m_0}{2} \rfloor + 4$.

This completes the proof. \square

Remark 3.6. We would like to remark that the case I_n implies $\chi \leq 1$. To see this, we compute the invariant $\chi(\mathcal{O}_X)$ under this situation. We have an induced fibration $f : X' \rightarrow B$ onto the smooth rational curve B . A general fiber S of f is a nonsingular projective surface of general type with $p_g(S) = 0$. Because $\chi(\mathcal{O}_S) > 0$, we see $q(S) = 0$. This means

$f_*\omega_{X'} = 0$ and $R^1f_*\omega_{X'} = 0$ since they are both torsion free. Thus we get by 2.4 the following formulae:

$$h^2(\mathcal{O}_X) = h^2(\mathcal{O}_{X'}) = h^1(f_*\omega_{X'}) + h^0(R^1f_*\omega_{X'}) = 0;$$

$$q(X) = q(X') = g(B) + h^1(R^1f_*\omega_{X'}) = 0.$$

So we see $\chi(\mathcal{O}_X) = 1 - q(X) + h^2(\mathcal{O}_X) - p_g(X) \leq 1$.

By a result of the first author and C. D. Hacon [4], $P_m > 0$ for all $m \geq 2$ if $q(V) > 0$. Hence $m_1 \leq 2$ for irregular varieties of general type. It follows that, for a threefold V with $\chi(\mathcal{O}_V) \leq 0$, one has $m_1 \leq 2$. We summarize the non-vanishing property as the following:

Corollary 3.7. *Let V be a nonsingular projective 3-fold of general type. Then $m_1 \leq 2$ if $\chi(\mathcal{O}_V) \leq 0$. Suppose furthermore that $P_{m_0} \geq 2$ for some $m_0 > 0$, then $m_1 \leq 3m_0 + 4$. If $P_{m_0} \geq 2$ and $\chi(\mathcal{O}_V) > 1$, then $m_1 \leq 2m_0 + 3$.*

4. Baskets of singularities

We always consider minimal projective 3-folds of general type in this section.

4.1. Terminal quotient singularity and basket. By a *3-dimensional terminal quotient singularity Q of type $\frac{1}{r}(1, -1, b)$* , we mean a singularity which is analytically isomorphic to the quotient of $(\mathbb{C}^3, \mathcal{O})$ by a cyclic group action ε :

$$\varepsilon(x, y, z) = (\varepsilon x, \varepsilon^{-1}y, \varepsilon^b z)$$

where r is a positive integer, ε is a fixed r -th primitive root of 1, the integer b is coprime to r and $0 < b < r$.

4.2. Convention. By replacing ε with another primitive root of 1 and changing the ordering of coordinates, we may even assume that $b \leq \frac{r}{2}$.

A *basket \mathcal{B} of singularities* is a collection (permitting weights) of terminal quotient singularities of type $\frac{1}{r_i}(1, -1, b_i)$, $i \in I$ where I is a finite index set. A *single basket* means a single singularity Q of type $\frac{1}{r}(1, -1, b)$. For simplicity, we will always denote a single basket by (b, r) . So we will simply write a basket as:

$$\mathcal{B} := \{n_i \times (b_i, r_i) | i \in I, n_i \in \mathbb{Z}^+\}.$$

Definition 4.3. When an integer b is not coprime to another integer r , we still call the symbol (b, r) a *generalized single basket* though it doesn't mean anything at this moment. A *generalized basket* means a collection of single baskets and generalized single baskets.

4.4. Plurigenera. Let us recall Reid's plurigenus formula (cf. [27], p413) for a minimal 3-fold X of general type (with \mathbb{Q} -factorial terminal singularities):

there exists a “virtual” basket² $\mathcal{B}(X)$ of terminal quotient singularities such that, for all $m > 1$,

$$P_m(X) = \frac{1}{12}m(m-1)(2m-1)K_X^3 - (2m-1)\chi(\mathcal{O}_X) + l(m) \quad (4.1)$$

where the correction term $l(m)$ can be computed as:

$$l(m) := \sum_{Q \in \mathcal{B}(X)} l_Q(m) := \sum_{Q \in \mathcal{B}(X)} \sum_{j=1}^{m-1} \frac{j\overline{b_Q}(r_Q - j\overline{b_Q})}{2r_Q}$$

where the sum \sum_Q runs through all single baskets Q of $\mathcal{B}(X)$ with type $\frac{1}{r_Q}(1, -1, b_Q)$ and $\overline{jb_Q}$ means the smallest residue of $jb_Q \bmod r_Q$.

We are going to analyze the above formula and Reid’s virtual basket $\mathcal{B}(X)$.

4.5. Invariants of baskets. Given a generalized single basket (b, r) (b not necessarily coprime to r) with $b \leq \frac{r}{2}$ and a fixed integer $n > 0$. Let $i := \lfloor \frac{bn}{r} \rfloor$. Then $\frac{i+1}{n} > \frac{b}{r} \geq \frac{i}{n}$. We define

$$\Delta_{b,r}^n := ibn - \frac{(i^2 + i)}{2}r.$$

One can see that $\Delta_{b,r}^n$ is a non-negative integer. For a generalized basket $B = \{(b_i, r_i) | i \in I\}$ and a fixed $n > 0$, we define $\Delta^n(B) := \sum_{i \in I} \Delta_{b_i, r_i}^n$.

By definition, $\Delta^2(B) = 0$ for any basket B . By a direct calculation, one gets the following relation:

$$\frac{j\overline{b_i}(r_i - j\overline{b_i})}{2r_i} - \frac{jb_i(r_i - jb_i)}{2r_i} = \Delta_{b_i, r_i}^j$$

for all $j > 0$. Define $\sigma(B) := \sum_{i \in I} b_i$ and $\sigma'(B) := \sum_{i \in I} \frac{b_i^2}{r_i}$.

4.6. Plurigenera in terms of Δ^m . We can rewrite Reid’s plurigenus formula as the following, where we take $B = \mathcal{B}(X)$ and $\Delta^m = \Delta^m(B)$:

$$(\partial) \begin{cases} P_2 &= \frac{1}{2}K_X^3 - 3\chi + \frac{1}{2}\sigma - \frac{1}{2}\sigma', \\ P_3 - P_2 &= \frac{1}{2}K_X^3 - 2\chi + \frac{1}{2}\sigma - \frac{1}{2}\sigma', \\ P_{m+1} - P_m &= \frac{m^2}{2}K_X^3 - 2\chi + \frac{m}{2}\sigma - \frac{m^2}{2}\sigma' + \Delta^m, \text{ for } m \geq 3. \end{cases}$$

4.7. Packing. Next we define a notion of “packing”. Given a generalized basket

$$B = \{(b_1, r_1), (b_2, r_2), \dots, (b_k, r_k)\},$$

we call the basket

$$B' := \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \dots, (b_k, r_k)\}$$

a packing of B , written as $B \succ B'$. If furthermore $b_1r_2 - b_2r_1 = 1$, we call $B \succ B'$ a *convenient packing*.

²Iano-Fletcher [17] has shown that Reid’s virtual basket $\mathcal{B}(X)$ is uniquely determined by X .

We have the following:

Lemma 4.8. *Let $B \succ B'$ be any packing between generalized baskets. Keep the same notations as above. Then:*

- (1) $\Delta^n(B) \geq \Delta^n(B')$ for all $n \geq 2$;
- (2) the equality in (1) holds if and only if $\frac{i}{n} \leq \frac{b_1}{r_1}, \frac{b_2}{r_2} \leq \frac{i+1}{n}$ for some i ;
- (3) $\sigma(B') = \sigma(B)$ and $\sigma'(B) = \sigma'(B') + \frac{(r_1 b_2 - r_2 b_1)^2}{r_1 r_2 (r_1 + r_2)} \geq \sigma'(B')$. Thus equality holds only when $\frac{b_1}{r_1} = \frac{b_2}{r_2}$.

Proof. First, if $\frac{i}{n} \leq \frac{b_1}{r_1}, \frac{b_2}{r_2} \leq \frac{i+1}{n}$ for some i , then a direct calculation shows $\Delta^n(B) = \Delta^n(B')$.

Suppose, for some $i > j$,

$$\frac{i+1}{n} > \frac{b_2}{r_2} \geq \frac{i}{n} \geq \frac{j+1}{n} > \frac{b_1}{r_1} \geq \frac{j}{n}$$

and $\frac{j_1+1}{n} > \frac{b_1+b_2}{r_1+r_2} \geq \frac{j_1}{n}$ for some $j_1 \in [j, i]$. Then

$$\begin{aligned} \Delta_{b_1+b_2, r_1+r_2}^n &= j_1 n (b_1 + b_2) - \frac{1}{2} (j_1^2 + j_1) (r_1 + r_2) \\ &= \Delta_{b_2, r_2}^n + \Delta_{b_1, r_1}^n + \nabla_2 + \nabla_1, \end{aligned}$$

where $\nabla_2 = (j_1 - i) n b_2 + \frac{1}{2} (i^2 + i - j_1^2 - j_1) r_2$ and $\nabla_1 = (j_1 - j) n b_1 + \frac{1}{2} (j^2 + j - j_1^2 - j_1) r_1$. Now since $n b_2 \geq i r_2$, one gets

$$\nabla_2 \leq \frac{1}{2} (i - j_1) (j_1 + 1 - i) r_2.$$

When $j_1 = i$, $\nabla_2 = 0$; when $j_1 = i - 1$, $\nabla_2 = -n b_1 + i r_2 \leq 0$; when $j_1 < i - 1$, $\nabla_2 < 0$.

Similarly the relation $n b_1 < (j + 1) r_1$ implies

$$\nabla_1 \leq \frac{1}{2} (j_1 - j) (j + 1 - j_1) r_1.$$

When $j_1 = j$, $\nabla_1 = 0$; when $j_1 = j + 1$, $\nabla_1 = n b_1 - (j + 1) r_1 < 0$; when $j_1 > j + 1$, $\nabla_1 < 0$.

Thus in any case, we see $\Delta^n(B) \geq \Delta^n(B')$, which implies (1). Furthermore we see $\Delta^n(B) = \Delta^n(B') = 0$ if and only if $\nabla_2 = \nabla_1$, if and only if $j_1 = j$ and $i = j_1 + 1 = j + 1$. We have proved (2).

The inequality (3) is obtained by a direct calculation. \square

Corollary 4.9. *If $B = \{m \times (b, r) \mid b \leq \frac{r}{2}, b \text{ coprime to } r\}$ and $B' = \{(mb, mr)\}$ for an integer $m > 1$, then*

- (i) $\sigma(B') = \sigma(B)$; $\sigma'(B') = \sigma'(B)$;
- (ii) $\Delta^n(B') = \Delta^n(B)$ for any $n > 0$.

Proof. This can be obtained by the definition of σ and Lemma 4.8. \square

Remark 4.10. The additive properties in Corollary 4.9 allow us to view the generalized single basket (mb, mr) as a basket $\{m \times (b, r)\}$.

Besides, a convenient packing has the following basic properties:

Lemma 4.11. *Let $B \succ B'$ be a convenient packing as in 4.7, i.e. $b_1 r_2 - b_2 r_1 = 1$. Then $\Delta_{b_1+b_2, r_1+r_2}^{r_1+r_2} = \Delta_{b_1, r_1}^{r_1+r_2} + \Delta_{b_2, r_2}^{r_1+r_2} - 1$.*

Proof. When $b_1 r_2 - b_2 r_1 = 1$, since $r_1 > 1, r_2 > 1$, one has

$$\frac{b_1 + b_2 + 1}{r_1 + r_2} > \frac{b_1}{r_1} > \frac{b_1 + b_2}{r_1 + r_2} > \frac{b_2}{r_2} > \frac{b_1 + b_2 - 1}{r_1 + r_2}.$$

We set $n = r_1 + r_2$. A direct calculation gives the equality

$$\Delta_{b_1+b_2, r_1+r_2}^n = \Delta_{b_1, r_1}^n + \Delta_{b_2, r_2}^n - 1.$$

□

4.12. Initial basket and limiting process. Given a basket $B = \{(b_i, r_i) \mid i \in I, b_i \text{ coprime to } r_i, b_i \leq \frac{r_i}{2}\}$ with I a finite set, we define a sequence of baskets $\{\mathcal{B}^{(n)}(B)\}$.

Take a set $S^{(0)} := \{\frac{1}{n}\}_{n \geq 2}$. For any single basket $B_i = (b_i, r_i) \in B$, we can find a unique $n > 0$ such that $\frac{1}{n} > \frac{b_i}{r_i} \geq \frac{1}{n+1}$. The single basket (b_i, r_i) can be regarded as successive packings via finite steps beginning from the basket $B_i^{(0)} := \{(nb_i + b_i - r_i) \times (1, n), (r_i - nb_i) \times (1, n+1)\}$. Adding up those $B_i^{(0)}$, one obtains the basket $\mathcal{B}^{(0)}(B) = \{n_{1,2} \times (1, 2), n_{1,3} \times (1, 3), \dots, n_{1,r} \times (1, r)\}$, called *the initial basket* of B . Clearly $\mathcal{B}^{(0)}(B) \succ B$. Defined in this way, $\mathcal{B}^{(0)}(B)$ is uniquely determined by the given basket B .

We begin to construct related baskets $\{\mathcal{B}^{(n)}(B)\}$ for $n \geq 1$. Consider the sets $S^{(1)} = S^{(2)} = S^{(3)} = S^{(4)} = S^{(0)}$ and

$$S^{(5)} := S^{(0)} \cup \{\frac{2}{5}\}$$

and inductively, $S^{(n)} = S^{(n-1)} \cup \{\frac{j}{n}\}_{j=2, \dots, \lfloor \frac{n}{2} \rfloor}$. Reordering elements in $S^{(n)}$ and writing $S^{(n)} = \{w_i^{(n)}\}_{i \in I}$ such that $w_i^{(n)} > w_{i+1}^{(n)}$ for all i , then we see that the interval $(0, \frac{1}{2}] = \cup_i [w_{i+1}^{(n)}, w_i^{(n)}]$. Note that $w_i^{(n)} = \frac{q_i}{p_i}$ with p_i coprime to q_i and $p_i \leq n$ unless $w_i^{(n)} = \frac{1}{m}$ for some $m > n$. First we prove the following:

Claim A. $p_{i+1}q_i - p_iq_{i+1} = 1$ for any two endpoints of $[w_{i+1}^{(n)}, w_i^{(n)}] = [\frac{q_{i+1}}{p_{i+1}}, \frac{q_i}{p_i}]$.

Proof. We can prove this inductively. Suppose that this property holds for $S^{(n-1)}$. Now, for any $\frac{j}{n} \in S^{(n)} - S^{(n-1)}$, $\frac{j}{n} \in [w_{i+1}^{(n-1)}, w_i^{(n-1)}]$ for some i . Thus $\frac{q_{i+1}}{p_{i+1}} < \frac{j}{n} < \frac{q_i}{p_i}$. If $p_i \geq n$, then $\frac{q_i}{p_i} = \frac{1}{m}$ and $\frac{q_{i+1}}{p_{i+1}} = \frac{1}{m+1}$ for some $m \geq n$ which contradicts to $\frac{j}{n} < \frac{q_i}{p_i}$. Therefore, we must have $p_i < n$. Then we consider $\frac{j-q_i}{n-p_i}$ and it's easy to see that

$$\frac{q_{i+1}}{p_{i+1}} \leq \frac{j - q_i}{n - p_i} < \frac{j}{n} < \frac{q_i}{p_i}.$$

Clearly, $\frac{j-q_i}{n-p_i} \in S^{(n-1)}$ and hence $\frac{j-q_i}{n-p_i} = \frac{q_{i+1}}{p_{i+1}}$. It follows that $n = p_i + \alpha p_{i+1}$, $j = q_i + \alpha q_{i+1}$ for some integer $\alpha > 0$.

If $\alpha \geq 2$, then $\frac{q_{i+1}}{p_{i+1}} < \frac{q_i + (\alpha-1)q_{i+1}}{p_i + (\alpha-1)p_{i+1}} < \frac{q_i}{p_i}$, and $\frac{q_i + (\alpha-1)q_{i+1}}{p_i + (\alpha-1)p_{i+1}} \in S^{(n-1)}$, which is absurd. Thus $\alpha = 1$ and then $n = p_i + p_{i+1}$, $j = q_i + q_{i+1}$. It's then clear that $\frac{j}{n}$ is the only element of $S^{(n)}$ inside the interval $[\frac{q_{i+1}}{p_{i+1}}, \frac{q_i}{p_i}]$. Moreover, $jp_{i+1} - nq_{i+1} = 1$, $nq_i - jp_i = 1$. This completes the proof of the claim. \square

Now for a single basket $B_i = (b_i, r_i) \in B$, if $\frac{b_i}{r_i} \in S^{(n)}$, then we set $B_i^{(n)} := \{(b_i, r_i)\}$. If $\frac{b_i}{r_i} \notin S^{(n)}$, then $\frac{q_1}{p_1} < \frac{b_i}{r_i} < \frac{q_2}{p_2}$ for some interval $[\frac{q_1}{p_1}, \frac{q_2}{p_2}]$ due to $S^{(n)}$. In this situation, we can unpack (b_i, r_i) to $B_i^{(n)} := \{(r_i q_2 - b_i p_2) \times (q_1, p_1), (-r_i q_1 + b_i p_1) \times (q_2, p_2)\}$. Adding up those $B_i^{(n)}$, we get a new basket $\mathcal{B}^{(n)}(B)$. $\mathcal{B}^{(n)}(B)$ is uniquely defined according to our construction and $\mathcal{B}^{(n)}(B) \succ B$ for all n .

Claim B. $\mathcal{B}^{(n-1)}(B) = \mathcal{B}^{(n-1)}(\mathcal{B}^{(n)}(B)) \succ \mathcal{B}^{(n)}(B)$ for all $n \geq 1$.

Proof. It's clear that $\mathcal{B}^{(n-1)}(\mathcal{B}^{(n)}(B)) \succ \mathcal{B}^{(n)}(B)$. Thus it suffices to show the first equality. By the definition of $\mathcal{B}^{(n)}$, we only need to prove for each single basket $B_i = (b_i, r_i) \in B$ and $n \geq 5$.

If $\frac{b_i}{r_i} \in S^{(n-1)} \subset S^{(n)}$, then there is nothing to prove since the equality follows from the definition of $\mathcal{B}^{(n)}$ and $\mathcal{B}^{(n-1)}$.

If $\frac{b_i}{r_i} \in S^{(n)} - S^{(n-1)}$, then this is also clear since $\mathcal{B}^{(n)}(B_i) = B_i$.

Suppose finally that $\frac{b_i}{r_i} \notin S^{(n)}$. Then $\frac{q_1}{p_1} < \frac{b_i}{r_i} < \frac{q_2}{p_2}$ for some $\frac{q_1}{p_1} = w_{i+1}^{(n)}$ and $\frac{q_2}{p_2} = w_i^{(n)}$.

Subcase (i). If both of $\frac{q_1}{p_1}$, $\frac{q_2}{p_2}$ are in $S^{(n)} - S^{(n-1)}$, then $p_1 = p_2 = n$ and hence $p_1 q_2 - p_2 q_1 \neq 1$, a contradiction to Claim A.

Subcase (ii). If both $\frac{q_1}{p_1}$ and $\frac{q_2}{p_2}$ are in $S^{(n-1)}$, then by definition

$$\mathcal{B}^{(n-1)}(B_i) = \mathcal{B}^{(n)}(B_i) = \mathcal{B}^{(n-1)}(\mathcal{B}^{(n)}(B_i)).$$

Subcase (iii). We are left to consider the situation that one of the $\frac{q_1}{p_1}$, $\frac{q_2}{p_2}$ is in $S^{(n-1)}$, but another one is in $S^{(n)} - S^{(n-1)}$. Let us assume, for example, $\frac{q_1}{p_1} = w_{j+1}^{(n-1)} \in S^{(n-1)}$. Then $\frac{q_2}{p_2} < w_j^{(n-1)} = \frac{q}{p} \in S^{(n-1)}$. The proof for the other case is similar. Notice that by the proof of Claim A, we have $q_2 = q_1 + q$, $p_2 = p_1 + p$. By definition,

$$\begin{aligned} \mathcal{B}^{(n)}(B_i) &= \{(r_i q_2 - b_i p_2) \times (q_1, p_1), (-r_i q_1 + b_i p_1) \times (q_2, p_2)\}, \\ \mathcal{B}^{(n-1)}(B_i) &= \{(r_i q - b_i p) \times (q_1, p_1), (-r_i q_1 + b_i p_1) \times (q, p)\}. \end{aligned}$$

Since $\mathcal{B}^{(n-1)}(q_2, p_2) = \{(q_1, p_1), (q, p)\}$, we get the following by computation:

$$\begin{aligned} \mathcal{B}^{(n-1)}(\mathcal{B}^{(n)}(B_i)) &= \{(r_i q_2 - b_i p_2) \times (q_1, p_1)\} + \{(-r_i q_1 + b_i p_1) \times (q_1, p_1), \\ &\quad (-r_i q_1 + b_i p_1) \times (q, p)\} \\ &= \{(r_i q - b_i p) \times (q_1, p_1), (-r_i q_1 + b_i p_1) \times (q, p)\}. \end{aligned}$$

So we can see $\mathcal{B}^{(n-1)}(B_i) = \mathcal{B}^{(n-1)}(\mathcal{B}^{(n)}(B_i))$. We are done. \square

By Claim B, we have obtained a chain $\{\mathcal{B}^{(n)}(B)\}$ of baskets with the following relation:

$$\mathcal{B}^{(0)}(B) = \dots = \mathcal{B}^{(4)}(B) \succ \mathcal{B}^{(5)}(B) \succ \dots \succ \mathcal{B}^{(n)}(B) \succ \dots \succ B. \quad (4.2)$$

Clearly $B = \mathcal{B}^{(n)}(B)$ for some $n \gg 0$ for a given finite basket B . Thus, in some sense, B can be realized as the limit of the sequence $\{\mathcal{B}^{(n)}(B)\}$.

Another direct consequence of Claim B is the property:

$$\mathcal{B}^{(i)}(\mathcal{B}^{(j)}(B)) = \mathcal{B}^{(i)}(B) \quad (4.3)$$

for $i \leq j$.

4.13. The quantity $\epsilon_n(B)$. Now let us consider the step $\mathcal{B}^{(n-1)}(B) \succ \mathcal{B}^{(n)}(B)$. For an element $w \in S^{(n)}$, let $m(w)$ be the number of basket (b, r) in $\mathcal{B}^{(n)}(B)$ with b coprime to r and $\frac{b}{r} = w$. Thus we can write $\mathcal{B}^{(n)}(B) = \{m(w) \times (b, r)\}_{w=\frac{b}{r} \in S^{(n)}}$.

Suppose that $S^{(n)} - S^{(n-1)} = \{\frac{j_s}{n}\}_{s=1, \dots, t}$. We have $w_{i_s}^{(n-1)} = \frac{q_{i_s}}{p_{i_s}} > \frac{j_s}{n} > w_{i_s+1}^{(n-1)} = \frac{q_{i_s+1}}{p_{i_s+1}}$ for some i_s . We remark that by the proof of Claim A, $j_s = q_{i_s} + q_{i_s+1}$, $n = p_{i_s} + p_{i_s+1}$. Since $\mathcal{B}^{(n-1)}(B) = \mathcal{B}^{(n-1)}(\mathcal{B}^{(n)}(B))$ by Claim B, we may write

$$\mathcal{B}^{(n)}(B) = \{m(w) \times (b, r)\}_{w=\frac{b}{r} \in S^{(n-1)}} + \{m(\frac{j_s}{n}) \times (j_s, n)\}_{\frac{j_s}{n}},$$

where "+" means collecting baskets of the same type. Then

$$\begin{aligned} \mathcal{B}^{(n-1)}(B) = & \{m(w) \times (b, r)\}_{w=\frac{b}{r} \in S^{(n-1)}} + \{m(\frac{j_s}{n}) \times (q_{i_s}, p_{i_s}), \\ & m(\frac{j_s}{n}) \times (q_{i_s+1}, p_{i_s+1})\}_{\frac{j_s}{n}}. \end{aligned}$$

We define $\epsilon_n(B) := \sum_{s=1}^t m(\frac{j_s}{n})$, which is the number of type (j_s, n) baskets with $\frac{j_s}{n} \notin S^{(n-1)}$. In other words, $\epsilon_n(B)$ counts the number of those single baskets (j_s, n) in $\mathcal{B}^{(n)}(B)$ with $(j_s, n) = 1$ and $j_s > 1$. This is going to be an important quantity in our arguments.

4.14. Notation. When no confusion is likely, we will simply write $B^{(n)}$ for $\mathcal{B}^{(n)}(B)$.

Lemma 4.15. *For the sequence $\{B^{(n)}\}$, the following statements are true:*

- (i) $\Delta^j(B^{(0)}) = \Delta^j(B)$ for $j = 3, 4$;
- (ii) $\Delta^j(B^{(n-1)}) = \Delta^j(B^{(n)})$ for all $j < n$;
- (iii) $\Delta^n(B^{(n-1)}) = \Delta^n(B^{(n)}) + \epsilon_n(B)$.
- (iv) $\Delta^n(B^{(n)}) = \Delta^n(B)$.

Proof. From $B^{(0)}$ to B , via $B^{(n)}$, the whole process can be realized through a composition of finite number of convenient packings. Each step is of the form $\{(q_1, p_1), (q_2, p_2)\} \succ \{(q_1 + q_2, p_1 + p_2)\}$. Notice that either $\frac{q_1}{p_1}, \frac{q_2}{p_2} \leq \frac{1}{3}$ or $\frac{q_1}{p_1}, \frac{q_2}{p_2} \geq \frac{1}{3}$. By Lemma 4.8(2), one gets $\Delta^3(B^{(0)}) = \Delta^3(B)$. The proof for Δ^4 is similar.

We consider now the step $B^{(n-1)} \succ B^{(n)}$. A direct computation shows that

$$\begin{aligned} & \Delta^n(B^{(n-1)}) - \Delta^n(B^{(n)}) \\ &= \sum_{s=1}^t m\left(\frac{j_s}{n}\right) (\Delta_{q_{i_s}, p_{i_s}}^n + \Delta_{q_{i_s+1}, p_{i_s+1}}^n - \Delta_{j_s, n}^n) \\ &= \sum_{s=1}^t m\left(\frac{j_s}{n}\right) (\Delta_{q_{i_s}, p_{i_s}}^n + \Delta_{q_{i_s+1}, p_{i_s+1}}^n - \Delta_{q_{i_s}+q_{i_s+1}, p_{i_s}+p_{i_s+1}}^n) \\ &= \sum_{s=1}^t m\left(\frac{j_s}{n}\right) \\ &= \epsilon_n(B). \end{aligned}$$

Finally, for any $j < n$, and suppose that $\frac{k+1}{j} \geq \frac{q_{i_s}}{p_{i_s}} = w_{i_s}^{(n-1)} > \frac{k}{j}$ for some k . Then $\frac{k+1}{j} \in S^{(n-1)}$ by definition. Thus $\frac{q_{i_s+1}}{p_{i_s+1}} = w_{i_s+1}^{(n-1)} \geq \frac{k}{j}$. By Lemma 4.8, we have

$$\Delta_{q_{i_s}, p_{i_s}}^j + \Delta_{q_{i_s+1}, p_{i_s+1}}^j = \Delta_{q_{i_s}+q_{i_s+1}, p_{i_s}+p_{i_s+1}}^j.$$

The last statement is due to (ii) and the fact that $B = B^{(n)}$ for a sufficiently large n . This completes the proof. \square

Let us go back to the sequence (4.2)

$$B^{(0)} \succ B^{(5)} \succ \dots \succ B^{(n)} \succ \dots \succ B.$$

We see that $\Delta^j(B^{(n)}) = \Delta^j(B)$ for all $j < n$. Thus $B^{(n)}$ can be viewed as an approximation of B of degree n . Also each approximation step $B^{(n-1)} \succ B^{(n)}$ is nothing but the convenient packings of ϵ_n pairs of baskets of type (b, n) with b coprime to n , $b \leq \frac{n}{2}$ and $b > 1$.

The whole strategy of our method is that, given a basket B , we can almost determine $B^{(n)}$ (for small n) in terms of P_m and $\chi(\mathcal{O}_X)$. Then we are able to recover B from $B^{(n)}$ because there are only finitely many baskets dominated by $B^{(n)}$. Finally we check whether those recovered baskets satisfy some geometric constraints. This works very effectively as seen in next sections.

5. Formal baskets

Given a minimal 3-fold X of general type, there is an associated basket $B := \mathcal{B}(X)$. In Section 4, we have defined the invariants $\sigma(B)$, $\sigma'(B)$ and $\Delta^m(B)$ which satisfy the equalities (∂) . The main purpose of our studying baskets is to classify $\mathcal{B}(X)$. To this end, we will study in a slightly general way. From now on within this section, we assume

$$B = \{(b_i, r_i) | i = 1, \dots, t; b_i, r_i > 0; b_i \leq \frac{r_i}{2}\}.$$

Definition 5.1. Any basket B as above is called a *normal basket*.

Notice that, by the equalities (∂) , all P_m are determined by $\sigma, \sigma' - K^3, \chi, \Delta^j$ for all $j < m$. These, in turn, are determined by B, χ and P_2 by virtue of the first equality in (∂) . This leads us to consider a more general setting.

Definition 5.2. Assume that B is a normal basket, $\tilde{\chi}$ and \tilde{P}_2 (≥ 0) are integers. The triple $\mathbf{B} := \{B, \tilde{\chi}, \tilde{P}_2\}$ is called a *formal basket*.

First we define $P_2(\mathbf{B}) := \tilde{P}_2$,

$$P_3(\mathbf{B}) := -\sigma(B) + 10\tilde{\chi} + 5\tilde{P}_2$$

and the volume

$$\begin{aligned} K^3(\mathbf{B}) &:= \sigma'(B) - 4\tilde{\chi} - 3\tilde{P}_2 + P_3(\mathbf{B}) \\ &= -\sigma + \sigma' + 6\tilde{\chi} + 2\tilde{P}_2. \end{aligned}$$

For $m \geq 4$, the plurigenus $P_m(\mathbf{B})$ is defined inductively by

$$P_{m+1}(\mathbf{B}) - P_m(\mathbf{B}) := \frac{m^2}{2}(K^3(\mathbf{B}) - \sigma'(B)) - 2\tilde{\chi} + \frac{m}{2}\sigma(B) + \Delta^m(B). \quad (5.1)$$

Clearly, by definition, $P_m(\mathbf{B})$ is an integer for all $m \geq 4$ because $K^3(\mathbf{B}) - \sigma'(B) = -4\tilde{\chi} - 3\tilde{P}_2 + P_3(\mathbf{B})$ and $\sigma = 10\tilde{\chi} + 5\tilde{P}_2 - P_3(\mathbf{B})$ have the same parity. Sometimes we even use the notations $K^3(B)$ and $P_m(B)$ to denote the volume and plurigenus.

Given a minimal 3-fold X , one can associate to X a triple $\mathbf{B}(X) := \{B, \tilde{\chi}, \tilde{P}_2\}$ where $B = \mathcal{B}(X)$, $\tilde{\chi} = \chi(\mathcal{O}_X)$ and $\tilde{P}_2 = P_2(X)$. It's clear that such a triple is a formal basket.

Definition 5.3. A formal basket \mathbf{B} is said to be *positive* if $K^3(\mathbf{B}) > 0$. \mathbf{B} is called *admissible* if $P_m(\mathbf{B}) \geq 0$ for all $m \geq 2$. \mathbf{B} is said to be *geometric* if $\mathbf{B} := \mathbf{B}(X)$ for some minimal 3-fold X .

Let X be a minimal 3-fold of general type. Because $\chi(\mathcal{O}_X)$ is an integer, $K^3(\mathbf{B}(X)) = K_X^3 > 0$ and $P_m(\mathbf{B}(X)) = P_m(X) \geq 0$ for all $m \geq 2$. It's clear that the formal basket $\mathbf{B}(X)$ is admissible and positive. Therefore it is sufficient for us to classify admissible and positive formal baskets. Indeed, it's enough to consider admissible and positive formal basket with some additionally imposed geometric conditions.

The point of view of packing baskets allows us to classify admissible baskets in an effective way. In what follows, we only consider packings in the approximation consideration as in 4.12. Thus all packings are convenient unless otherwise stated.

5.4. Notations. We assume that $\mathbf{B} = \{B, \tilde{\chi}, \tilde{P}_2\}$ is an admissible and positive formal basket. For simplicity, we denote $P_m(\mathbf{B})$ by \tilde{P}_m for all $m \geq 4$. Also denote $K^3(\mathbf{B})$ by \tilde{K}^3 , $\sigma = \sigma(B)$, $\sigma' = \sigma'(B)$ and $\Delta^m = \Delta^m(B)$.

In what follows, we would like to classify positive admissible formal baskets with given datum $(\tilde{\chi}, \tilde{P}_2, \tilde{P}_3, \dots, \tilde{P}_m)$.

First of all, by the definition of \tilde{K}^3 and \tilde{P}_m , we get:

$$\begin{aligned}
\tau := \sigma' - \tilde{K}^3 &= 4\tilde{\chi} + 3\tilde{P}_2 - \tilde{P}_3, \\
\sigma &= 10\tilde{\chi} + 5\tilde{P}_2 - \tilde{P}_3 \\
\Delta^3 &= 5\tilde{\chi} + 6\tilde{P}_2 - 4\tilde{P}_3 + \tilde{P}_4 \\
\Delta^4 &= 14\tilde{\chi} + 14\tilde{P}_2 - 6\tilde{P}_3 - \tilde{P}_4 + \tilde{P}_5 \\
\Delta^5 &= 27\tilde{\chi} + 25\tilde{P}_2 - 10\tilde{P}_3 - \tilde{P}_5 + \tilde{P}_6 \\
\Delta^6 &= 44\tilde{\chi} + 39\tilde{P}_2 - 15\tilde{P}_3 - \tilde{P}_6 + \tilde{P}_7 \\
\Delta^7 &= 65\tilde{\chi} + 56\tilde{P}_2 - 21\tilde{P}_3 - \tilde{P}_7 + \tilde{P}_8 \\
\Delta^8 &= 90\tilde{\chi} + 76\tilde{P}_2 - 28\tilde{P}_3 - \tilde{P}_8 + \tilde{P}_9 \\
\Delta^9 &= 119\tilde{\chi} + 99\tilde{P}_2 - 36\tilde{P}_3 - \tilde{P}_9 + \tilde{P}_{10} \\
\Delta^{10} &= 152\tilde{\chi} + 125\tilde{P}_2 - 45\tilde{P}_3 - \tilde{P}_{10} + \tilde{P}_{11} \\
\Delta^{11} &= 189\tilde{\chi} + 154\tilde{P}_2 - 55\tilde{P}_3 - \tilde{P}_{11} + \tilde{P}_{12} \\
\Delta^{12} &= 230\tilde{\chi} + 186\tilde{P}_2 - 66\tilde{P}_3 - \tilde{P}_{12} + \tilde{P}_{13}
\end{aligned}$$

Recall that $B^{(0)} = \{n_{1,2}^0 \times (1, 2), \dots, n_{1,r}^0 \times (1, r)\}$ is the initial basket of B . Then by Lemma 4.15 and the definition of $\sigma(B)$, we have

$$\begin{aligned}
\sigma(B) &= \sigma(B^{(0)}) = \sum n_{1,r}^0, \\
\Delta^3(B) &= \Delta^3(B^{(0)}) = n_{1,2}^0 \\
\Delta^4(B) &= \Delta^4(B^{(0)}) = 2n_{1,2}^0 + n_{1,3}^0
\end{aligned}$$

Therefore, the initial basket has the coefficients:

$$B^{(0)} \begin{cases} n_{1,2}^0 = 5\tilde{\chi} + 6\tilde{P}_2 - 4\tilde{P}_3 + \tilde{P}_4 \\ n_{1,3}^0 = 4\tilde{\chi} + 2\tilde{P}_2 + 2\tilde{P}_3 - 3\tilde{P}_4 + \tilde{P}_5 \\ n_{1,4}^0 = \tilde{\chi} - 3\tilde{P}_2 + \tilde{P}_3 + 2\tilde{P}_4 - \tilde{P}_5 - \sum_{r \geq 5} n_{1,r}^0 \\ n_{1,r}^0 = n_{1,r}^0, r \geq 5 \end{cases}$$

By Lemma 4.15, we see

$$\begin{aligned}
\epsilon_5 &:= \Delta^5(B^{(0)}) - \Delta^5(B) = 4n_{1,2}^0 + 2n_{1,3}^0 + n_{1,4}^0 - \Delta^5(B) \\
&= 2\tilde{\chi} - \tilde{P}_3 + 2\tilde{P}_5 - \tilde{P}_6 - \sigma_5,
\end{aligned}$$

where $\sigma_5 := \sum_{r \geq 5} n_{1,r}^0$. Thus we can write

$$B^{(5)} = \{n_{1,2}^5 \times (1, 2), n_{2,5}^5 \times (2, 5), n_{1,3}^5 \times (1, 3), n_{1,4}^5 \times (1, 4), n_{1,5}^5 \times (1, 5), \dots\}$$

with

$$B^{(5)} \begin{cases} n_{1,2}^5 = 3\tilde{\chi} + 6\tilde{P}_2 - 3\tilde{P}_3 + \tilde{P}_4 - 2\tilde{P}_5 + \tilde{P}_6 + \sigma_5, \\ n_{2,5}^5 = 2\tilde{\chi} - \tilde{P}_3 + 2\tilde{P}_5 - \tilde{P}_6 - \sigma_5 \\ n_{1,3}^5 = 2\tilde{\chi} + 2\tilde{P}_2 + 3\tilde{P}_3 - 3\tilde{P}_4 - \tilde{P}_5 + \tilde{P}_6 + \sigma_5, \\ n_{1,4}^5 = \tilde{\chi} - 3\tilde{P}_2 + \tilde{P}_3 + 2\tilde{P}_4 - \tilde{P}_5 - \sigma_5 \\ n_{1,r}^5 = n_{1,r}^0, r \geq 5 \end{cases}$$

noting that this is obtained by convenient packing $\{(1, 2), (1, 3)\} \succ \{(2, 5)\}$.

Clearly, $B^{(5)} = B^{(6)}$ by our construction. Thus by Lemma 4.15 we have $\Delta^6(B^{(5)}) = \Delta^6(B^{(6)}) = \Delta^6(B)$. Computation shows that

$$\begin{aligned}\Delta^6(B^{(5)}) &= 6n_{1,2}^5 + 9n_{2,5}^5 + 3n_{1,3}^5 + 2n_{1,4}^5 + n_{1,5}^5 \\ &= 44\tilde{\chi} + 36\tilde{P}_2 - 16\tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - \epsilon,\end{aligned}$$

where

$$\epsilon := n_{1,5}^0 + 2 \sum_{r \geq 6} n_{1,r}^0 = 2\sigma_5 - n_{1,5}^0.$$

So we see

$$\epsilon_6 := -3\tilde{P}_2 - \tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 + \tilde{P}_6 - \tilde{P}_7 - \epsilon = 0. \quad (5.2)$$

Next, we compute

$$\begin{aligned}\epsilon_7 : &= \Delta^7(B^{(6)}) - \Delta^7(B) = \Delta^7(B^{(5)}) - \Delta^7(B) \\ &= 9n_{1,2}^5 + 13n_{2,5}^5 + 5n_{1,3}^5 + 3n_{1,4}^5 + 2n_{1,5}^5 + n_{1,6}^5 - \Delta^7(B) \\ &= \tilde{\chi} - \tilde{P}_2 - \tilde{P}_3 + \tilde{P}_6 + \tilde{P}_7 - \tilde{P}_8 - 2\sigma_5 + 2n_{1,5}^0 + n_{1,6}^0.\end{aligned}$$

Since $S^{(7)} - S^{(6)} = \{\frac{2}{7}, \frac{3}{7}\}$, there are two ways of packing into basket of type $(b, 7)$. Let $\eta \geq 0$ be the number of packing $\{(1, 3), (1, 4)\} \succ \{(2, 7)\}$. Then $\epsilon_7 - \eta \geq 0$ is the number of packing $\{(1, 2), (2, 5)\} \succ \{(3, 7)\}$. Thus we can write $B^{(7)} = \{n_{b,r}^7 \times (b, r)\}_{\frac{b}{r} \in S^{(7)}}$ with

$$B^{(7)} \begin{cases} n_{1,2}^7 = 2\tilde{\chi} + 7\tilde{P}_2 - 2\tilde{P}_3 + \tilde{P}_4 - 2\tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + 3\sigma_5 - 2n_{1,5}^0 - n_{1,6}^0 + \eta \\ n_{3,7}^7 = \tilde{\chi} - \tilde{P}_2 - \tilde{P}_3 + \tilde{P}_6 + \tilde{P}_7 - \tilde{P}_8 - 2\sigma_5 + 2n_{1,5}^0 + n_{1,6}^0 - \eta \\ n_{2,5}^7 = \tilde{\chi} + \tilde{P}_2 + 2\tilde{P}_5 - 2\tilde{P}_6 - \tilde{P}_7 + \tilde{P}_8 + \sigma_5 - 2n_{1,5}^0 - n_{1,6}^0 + \eta \\ n_{1,3}^7 = 2\tilde{\chi} + 2\tilde{P}_2 + 3\tilde{P}_3 - 3\tilde{P}_4 - \tilde{P}_5 + \tilde{P}_6 + \sigma_5 - \eta \\ n_{2,7}^7 = \eta \\ n_{1,4}^7 = \tilde{\chi} - 3\tilde{P}_2 + \tilde{P}_3 + 2\tilde{P}_4 - \tilde{P}_5 - \sigma_5 - \eta \\ n_{1,r}^7 = n_{1,r}^0, r \geq 5 \end{cases}$$

From $B^{(7)}$, we can compute ϵ_8 and then $B^{(8)}$, and inductively $B^{(n)}$ for all $n \geq 9$. But notice that one can even compute ϵ_9 , ϵ_{10} and ϵ_{12} directly from $B^{(7)}$, thanks to Lemma 4.8.

To see this, let's consider $\epsilon_9 := \Delta^9(B^{(8)}) - \Delta^9(B)$ for example. Note that $B^{(7)} \succ B^{(8)}$ is obtained by some convenient packing into $\{(b, 8)\}$, which is $\{(3, 8)\}$. And every such packing, which is $\{(2, 5), (1, 3)\} \succ \{(3, 8)\}$, happens inside a closed interval $[\frac{3}{9}, \frac{4}{9}]$. Thus by Lemma 4.8(2), $\Delta^9(B^{(8)}) = \Delta^9(B^{(7)})$. Similarly we can see $\Delta^{10}(B^{(9)}) = \Delta^{10}(B^{(7)})$ and $\Delta^{12}(B^{(10)}) = \Delta^{12}(B^{(7)})$. Unfortunately, $B^{11}(B^{(10)}) \neq \Delta^{11}(B^{(7)})$.

In summary, we have:

$$\begin{aligned}
\Delta^8(B^{(7)}) &= 12n_{1,2}^7 + 30n_{3,7}^7 + 18n_{2,5}^7 + 7n_{1,3}^7 + 11n_{2,7}^7 + 4n_{1,4}^7 \\
&\quad + 3n_{1,5}^7 + 2n_{1,6}^7 + n_{1,7}^7 \\
&= 90\tilde{\chi} + 74\tilde{P}_2 - 29\tilde{P}_3 - \tilde{P}_4 + \tilde{P}_5 + \tilde{P}_6 - 3\sigma_5 \\
&\quad + 3n_{1,5}^0 + 2n_{1,6}^0 + n_{1,7}^0; \\
\Delta^9(B^{(8)}) &= \Delta^9(B^{(7)}) \\
&= 16n_{1,2}^7 + 39n_{3,7}^7 + 24n_{2,5}^7 + 9n_{1,3}^7 + 15n_{2,7}^7 + 6n_{1,4}^7 \\
&\quad + 4n_{1,5}^7 + 3n_{1,6}^7 + 2n_{1,7}^7 + n_{1,8}^7 \\
&= 119\tilde{\chi} + 97\tilde{P}_2 - 38\tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 - 3\sigma_5 + \eta \\
&\quad + 2n_{1,5}^0 + 2n_{1,6}^0 + 2n_{1,7}^0 + n_{1,8}^0; \\
\Delta^{10}(B^{(9)}) &= \Delta^{10}(B^{(8)}) = \Delta^{10}(B^{(7)}) \\
&= 20n_{1,2}^7 + 50n_{3,7}^7 + 30n_{2,5}^7 + 12n_{1,3}^7 + 19n_{2,7}^7 + 8n_{1,4}^7 \\
&\quad + 5n_{1,5}^7 + 4n_{1,6}^7 + 3n_{1,7}^7 + 2n_{1,8}^7 + n_{1,9}^7 \\
&= 152\tilde{\chi} + 120\tilde{P}_2 - 46\tilde{P}_3 + 2\tilde{P}_6 - 6\sigma_5 - \eta \\
&\quad + 5n_{1,5}^0 + 4n_{1,6}^0 + 3n_{1,7}^0 + 2n_{1,8}^0 + n_{1,9}^0; \\
\Delta^{12}(B^{(11)}) &= \Delta^{12}(B^{(10)}) = \dots = \Delta^{12}(B^{(7)}) \\
&= 30n_{1,2}^7 + 75n_{3,7}^7 + 46n_{2,5}^7 + 18n_{1,3}^7 + 30n_{2,7}^7 + 12n_{1,4}^7 \\
&\quad + 9n_{1,5}^7 + 6n_{1,6}^7 + 5n_{1,7}^7 + 4n_{1,8}^7 + 3n_{1,9}^7 + 2n_{1,10}^7 + n_{1,11}^7 \\
&= 229\tilde{\chi} + 181\tilde{P}_2 - 69\tilde{P}_3 + 2\tilde{P}_5 + \tilde{P}_6 - \tilde{P}_7 + \tilde{P}_8 - 8\sigma_5 + \eta \\
&\quad + 7n_{1,5}^0 + 5n_{1,6}^0 + 5n_{1,7}^0 + 4n_{1,8}^0 + 3n_{1,9}^0 + 2n_{1,10}^0 + n_{1,11}^0.
\end{aligned}$$

We thus have:

$$\begin{aligned}
\epsilon_8 &= -2\tilde{P}_2 - \tilde{P}_3 - \tilde{P}_4 + \tilde{P}_5 + \tilde{P}_6 + \tilde{P}_8 - \tilde{P}_9 - 3\sigma_5 \\
&\quad + 3n_{1,5}^0 + 2n_{1,6}^0 + n_{1,7}^0; \\
\epsilon_9 &= -2\tilde{P}_2 - 2\tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + \tilde{P}_9 - \tilde{P}_{10} - 3\sigma_5 + \eta \\
&\quad + 2n_{1,5}^0 + 2n_{1,6}^0 + 2n_{1,7}^0 + n_{1,8}^0; \\
\epsilon_{10} &= -5\tilde{P}_2 - \tilde{P}_3 + 2\tilde{P}_6 + \tilde{P}_{10} - \tilde{P}_{11} - 6\sigma_5 - \eta \\
&\quad + 5n_{1,5}^0 + 4n_{1,6}^0 + 3n_{1,7}^0 + 2n_{1,8}^0 + n_{1,9}^0; \\
\epsilon_{12} &= -\tilde{\chi} - 5\tilde{P}_2 - 3\tilde{P}_3 + 2\tilde{P}_5 + \tilde{P}_6 - \tilde{P}_7 + \tilde{P}_8 + \tilde{P}_{12} - \tilde{P}_{13} - 8\sigma_5 + \eta \\
&\quad + 7n_{1,5}^0 + 5n_{1,6}^0 + 5n_{1,7}^0 + 4n_{1,8}^0 + 3n_{1,9}^0 + 2n_{1,10}^0 + n_{1,11}^0.
\end{aligned}$$

Since both ϵ_{10} and ϵ_{12} are non-negative, we have $\epsilon_{10} + \epsilon_{12} \geq 0$. This gives rise to:

$$2\tilde{P}_5 + 3\tilde{P}_6 + \tilde{P}_8 + \tilde{P}_{10} + \tilde{P}_{12} \geq \tilde{\chi} + 10\tilde{P}_2 + 4\tilde{P}_3 + \tilde{P}_7 + \tilde{P}_{11} + \tilde{P}_{13} + R, \quad (5.3)$$

where

$$\begin{aligned}
R &:= 14\sigma_5 - 12n_{1,5}^0 - 9n_{1,6}^0 - 8n_{1,7}^0 - 6n_{1,8}^0 - 4n_{1,9}^0 - 2n_{1,10}^0 - n_{1,11}^0 \\
&= 2n_{1,5}^0 + 5n_{1,6}^0 + 6n_{1,7}^0 + 8n_{1,8}^0 + 10n_{1,9}^0 + 12n_{1,10}^0 + 13n_{1,11}^0 + 14 \sum_{r \geq 12} n_{1,r}^0.
\end{aligned}$$

The equation (5.2) and the inequality (5.3) will play important roles in the following classification.

In practice, we will frequently end up with situations satisfying the following assumption and then our computation will be comparatively simpler.

5.5. Assumption. $\tilde{P}_2 = 0$ and $n_{1,r}^0 = 0$ for all $r \geq 6$.

Under Assumption 5.5, we list our datum in details as follows. First,

$$\epsilon_7 = \tilde{\chi} - \tilde{P}_3 + \tilde{P}_6 + \tilde{P}_7 - \tilde{P}_8$$

and $B^{(7)} = \{n_{b,r}^7 \times (b, r)\}_{\frac{b}{r} \in S^{(7)}}$ has coefficients:

$$B^{(7)} \begin{cases} n_{1,2}^7 = 2\tilde{\chi} - 2\tilde{P}_3 + \tilde{P}_4 - 2\tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + n_{1,5}^0 + \eta \\ n_{3,7}^7 = \tilde{\chi} - \tilde{P}_3 + \tilde{P}_6 + \tilde{P}_7 - \tilde{P}_8 - \eta \\ n_{2,5}^7 = \tilde{\chi} + 2\tilde{P}_5 - 2\tilde{P}_6 - \tilde{P}_7 + \tilde{P}_8 - n_{1,5}^0 + \eta \\ n_{1,3}^7 = 2\tilde{\chi} + 3\tilde{P}_3 - 3\tilde{P}_4 - \tilde{P}_5 + \tilde{P}_6 + n_{1,5}^0 - \eta \\ n_{2,7}^7 = \eta \\ n_{1,4}^7 = \tilde{\chi} + \tilde{P}_3 + 2\tilde{P}_4 - \tilde{P}_5 - n_{1,5}^0 - \eta \\ n_{1,5}^7 = n_{1,5}^0. \end{cases}$$

We have already known

$$\epsilon_8 = -\tilde{P}_3 - \tilde{P}_4 + \tilde{P}_5 + \tilde{P}_6 + \tilde{P}_8 - \tilde{P}_9.$$

Thus, taking some convenient packing into account, $B^{(8)} = \{n_{b,r}^8 \times (b, r)\}_{\frac{b}{r} \in S^{(8)}}$ has the coefficients:

$$B^{(8)} \begin{cases} n_{1,2}^8 = 2\tilde{\chi} - 2\tilde{P}_3 + \tilde{P}_4 - 2\tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + n_{1,5}^0 + \eta \\ n_{3,7}^8 = \tilde{\chi} - \tilde{P}_3 + \tilde{P}_6 + \tilde{P}_7 - \tilde{P}_8 - \eta \\ n_{2,5}^8 = \tilde{\chi} + \tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - 3\tilde{P}_6 - \tilde{P}_7 + \tilde{P}_9 - n_{1,5}^0 + \eta \\ n_{3,8}^8 = -\tilde{P}_3 - \tilde{P}_4 + \tilde{P}_5 + \tilde{P}_6 + \tilde{P}_8 - \tilde{P}_9 \\ n_{1,3}^8 = 2\tilde{\chi} + 4\tilde{P}_3 - 2\tilde{P}_4 - 2\tilde{P}_5 - \tilde{P}_8 + \tilde{P}_9 + n_{1,5}^0 - \eta \\ n_{2,7}^8 = \eta \\ n_{1,4}^8 = \tilde{\chi} + \tilde{P}_3 + 2\tilde{P}_4 - \tilde{P}_5 - n_{1,5}^0 - \eta \\ n_{1,5}^8 = n_{1,5}^0. \end{cases}$$

We know that

$$\epsilon_9 = -2\tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + \tilde{P}_9 - \tilde{P}_{10} - n_{1,5}^0 + \eta.$$

Moreover $S^{(9)} - S^{(8)} = \{\frac{4}{9}, \frac{2}{9}\}$. Let ζ be the number of packings $\{(1, 2), (3, 7)\} \succ \{(4, 9)\}$, then the number of $\{(1, 4), (1, 5)\} \succ \{(2, 9)\}$

packings is $\epsilon_9 - \zeta$. We can get $B^{(9)}$ consisting of the following coefficients.

$$B^{(9)} \left\{ \begin{array}{l} n_{1,2}^9 = 2\tilde{\chi} - 2\tilde{P}_3 + \tilde{P}_4 - 2\tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + n_{1,5}^0 + \eta - \zeta \\ n_{4,9}^9 = \zeta \\ n_{3,7}^9 = \tilde{\chi} - \tilde{P}_3 + \tilde{P}_6 + \tilde{P}_7 - \tilde{P}_8 - \eta - \zeta \\ n_{2,5}^9 = \tilde{\chi} + \tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - 3\tilde{P}_6 - \tilde{P}_7 + \tilde{P}_9 - n_{1,5}^0 + \eta \\ n_{3,8}^9 = -\tilde{P}_3 - \tilde{P}_4 + \tilde{P}_5 + \tilde{P}_6 + \tilde{P}_8 - \tilde{P}_9 \\ n_{1,3}^9 = 2\tilde{\chi} + 4\tilde{P}_3 - 2\tilde{P}_4 - 2\tilde{P}_5 - \tilde{P}_8 + \tilde{P}_9 + n_{1,5}^0 - \eta \\ n_{2,7}^9 = \eta \\ n_{1,4}^9 = \tilde{\chi} + 3\tilde{P}_3 + \tilde{P}_4 - 2\tilde{P}_5 + \tilde{P}_7 - \tilde{P}_8 - \tilde{P}_9 + \tilde{P}_{10} - 2\eta + \zeta \\ n_{2,9}^9 = -2\tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + \tilde{P}_9 - \tilde{P}_{10} - n_{1,5}^0 + \eta - \zeta \\ n_{1,5}^9 = 2\tilde{P}_3 - \tilde{P}_4 - \tilde{P}_5 + \tilde{P}_7 - \tilde{P}_8 - \tilde{P}_9 + \tilde{P}_{10} + 2n_{1,5}^0 - \eta + \zeta \end{array} \right.$$

One has

$$\epsilon_{10} = -\tilde{P}_3 + 2\tilde{P}_6 + \tilde{P}_{10} - \tilde{P}_{11} - n_{1,5}^0 - \eta$$

and then $B^{(10)}$ consists of following coefficients:

$$B^{(10)} \left\{ \begin{array}{l} n_{1,2}^{10} = 2\tilde{\chi} - 2\tilde{P}_3 + \tilde{P}_4 - 2\tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + n_{1,5}^0 + \eta - \zeta \\ n_{4,9}^{10} = \zeta \\ n_{3,7}^{10} = \tilde{\chi} - \tilde{P}_3 + \tilde{P}_6 + \tilde{P}_7 - \tilde{P}_8 - \eta - \zeta \\ n_{2,5}^{10} = \tilde{\chi} + \tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - 3\tilde{P}_6 - \tilde{P}_7 + \tilde{P}_9 - n_{1,5}^0 + \eta \\ n_{3,8}^{10} = -\tilde{P}_3 - \tilde{P}_4 + \tilde{P}_5 + \tilde{P}_6 + \tilde{P}_8 - \tilde{P}_9 \\ n_{1,3}^{10} = 2\tilde{\chi} + 5\tilde{P}_3 - 2\tilde{P}_4 - 2\tilde{P}_5 - 2\tilde{P}_6 - \tilde{P}_8 + \tilde{P}_9 - \tilde{P}_{10} + \tilde{P}_{11} + 2n_{1,5}^0 \\ n_{3,10}^{10} = -\tilde{P}_3 + 2\tilde{P}_6 + \tilde{P}_{10} - \tilde{P}_{11} - n_{1,5}^0 - \eta \\ n_{2,7}^{10} = \tilde{P}_3 - 2\tilde{P}_6 - \tilde{P}_{10} + \tilde{P}_{11} + n_{1,5}^0 + 2\eta \\ n_{1,4}^{10} = \tilde{\chi} + 3\tilde{P}_3 + \tilde{P}_4 - 2\tilde{P}_5 + \tilde{P}_7 - \tilde{P}_8 - \tilde{P}_9 + \tilde{P}_{10} - 2\eta + \zeta \\ n_{2,9}^{10} = -2\tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + \tilde{P}_9 - \tilde{P}_{10} - n_{1,5}^0 + \eta - \zeta \\ n_{1,5}^{10} = 2\tilde{P}_3 - \tilde{P}_4 - \tilde{P}_5 + \tilde{P}_7 - \tilde{P}_8 - \tilde{P}_9 + \tilde{P}_{10} + 2n_{1,5}^0 - \eta + \zeta \end{array} \right.$$

By computing $\Delta^{11}(B^{(10)})$, we get

$$\epsilon_{11} = \tilde{\chi} - \tilde{P}_3 + \tilde{P}_4 - \tilde{P}_7 + \tilde{P}_9 + \tilde{P}_{11} - \tilde{P}_{12} - n_{1,5}^0 - \zeta.$$

Let α be the number of packing $\{(1, 2), (4, 9)\} \succ \{(5, 11)\}$ and β be the number of packing $\{(1, 3), (3, 8)\} \succ \{(4, 11)\}$. Then we get $B^{(11)}$

with

$$B^{(11)} \left\{ \begin{array}{l} n_{1,2}^{11} = 2\tilde{\chi} - 2\tilde{P}_3 + \tilde{P}_4 - 2\tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + n_{1,5}^0 + \eta - \zeta - \alpha \\ n_{5,11}^{11} = \alpha \\ n_{4,9}^{11} = \zeta - \alpha \\ n_{3,7}^{11} = \tilde{\chi} - \tilde{P}_3 + \tilde{P}_6 + \tilde{P}_7 - \tilde{P}_8 - \eta - \zeta \\ n_{2,5}^{11} = \tilde{\chi} + \tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - 3\tilde{P}_6 - \tilde{P}_7 + \tilde{P}_9 - n_{1,5}^0 + \eta \\ n_{3,8}^{11} = -\tilde{P}_3 - \tilde{P}_4 + \tilde{P}_5 + \tilde{P}_6 + \tilde{P}_8 - \tilde{P}_9 - \beta \\ n_{4,11}^{11} = \beta \\ n_{1,3}^{11} = 2\tilde{\chi} + 5\tilde{P}_3 - 2\tilde{P}_4 - 2\tilde{P}_5 - 2\tilde{P}_6 - \tilde{P}_8 + \tilde{P}_9 - \tilde{P}_{10} + \tilde{P}_{11} + 2n_{1,5}^0 - \beta \\ n_{3,10}^{11} = -\tilde{P}_3 + 2\tilde{P}_6 + \tilde{P}_{10} - \tilde{P}_{11} - n_{1,5}^0 - \eta \\ n_{2,7}^{11} = -\tilde{\chi} + 2\tilde{P}_3 - \tilde{P}_4 - 2\tilde{P}_6 + \tilde{P}_7 - \tilde{P}_9 - \tilde{P}_{10} + \tilde{P}_{12} + 2n_{1,5}^0 + 2\eta + \zeta + \alpha + \beta \\ n_{3,11}^{11} = \tilde{\chi} - \tilde{P}_3 + \tilde{P}_4 - \tilde{P}_7 + \tilde{P}_9 + \tilde{P}_{11} - \tilde{P}_{12} - n_{1,5}^0 - \zeta - \alpha - \beta \\ n_{1,4}^{11} = 4\tilde{P}_3 - 2\tilde{P}_5 + 2\tilde{P}_7 - \tilde{P}_8 - 2\tilde{P}_9 + \tilde{P}_{10} - \tilde{P}_{11} + \tilde{P}_{12} + n_{1,5}^0 - 2\eta + 2\zeta + \alpha + \beta \\ n_{2,9}^{11} = -2\tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + \tilde{P}_9 - \tilde{P}_{10} - n_{1,5}^0 + \eta - \zeta \\ n_{1,5}^{11} = 2\tilde{P}_3 - \tilde{P}_4 - \tilde{P}_5 + \tilde{P}_7 - \tilde{P}_8 - \tilde{P}_9 + \tilde{P}_{10} + 2n_{1,5}^0 - \eta + \zeta \end{array} \right.$$

Finally since

$$\epsilon_{12} = -\tilde{\chi} - 3\tilde{P}_3 + 2\tilde{P}_5 + \tilde{P}_6 - \tilde{P}_7 + \tilde{P}_8 + \tilde{P}_{12} - \tilde{P}_{13} - n_{1,5}^0 + \eta.$$

we get $B^{(12)}$ with

$$B^{(12)} \left\{ \begin{array}{l} n_{1,2}^{12} = 2\tilde{\chi} - 2\tilde{P}_3 + \tilde{P}_4 - 2\tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + n_{1,5}^0 + \eta - \zeta - \alpha \\ n_{5,11}^{12} = \alpha \\ n_{4,9}^{12} = \zeta - \alpha \\ n_{3,7}^{12} = 2\tilde{\chi} + 2\tilde{P}_3 - 2\tilde{P}_5 + 2\tilde{P}_7 - 2\tilde{P}_8 - \tilde{P}_{12} + \tilde{P}_{13} - 2\eta - \zeta + n_{1,5}^0 \\ n_{5,12}^{12} = -\tilde{\chi} - 3\tilde{P}_3 + 2\tilde{P}_5 + \tilde{P}_6 - \tilde{P}_7 + \tilde{P}_8 + \tilde{P}_{12} - \tilde{P}_{13} + \eta - n_{1,5}^0 \\ n_{2,5}^{12} = 2\tilde{\chi} + 4\tilde{P}_3 + \tilde{P}_4 - \tilde{P}_5 - 4\tilde{P}_6 - \tilde{P}_8 + \tilde{P}_9 - \tilde{P}_{12} + \tilde{P}_{13} \\ n_{3,8}^{12} = -\tilde{P}_3 - \tilde{P}_4 + \tilde{P}_5 + \tilde{P}_6 + \tilde{P}_8 - \tilde{P}_9 - \beta \\ n_{4,11}^{12} = \beta \\ n_{1,3}^{12} = 2\tilde{\chi} + 5\tilde{P}_3 - 2\tilde{P}_4 - 2\tilde{P}_5 - 2\tilde{P}_6 - \tilde{P}_8 + \tilde{P}_9 - \tilde{P}_{10} + \tilde{P}_{11} + 2n_{1,5}^0 - \beta \\ n_{3,10}^{12} = -\tilde{P}_3 + 2\tilde{P}_6 + \tilde{P}_{10} - \tilde{P}_{11} - n_{1,5}^0 - \eta \\ n_{2,7}^{12} = -\tilde{\chi} + 2\tilde{P}_3 - \tilde{P}_4 - 2\tilde{P}_6 + \tilde{P}_7 - \tilde{P}_9 - \tilde{P}_{10} + \tilde{P}_{12} + 2n_{1,5}^0 + 2\eta + \zeta + \alpha + \beta \\ n_{3,11}^{12} = \tilde{\chi} - \tilde{P}_3 + \tilde{P}_4 - \tilde{P}_7 + \tilde{P}_9 + \tilde{P}_{11} - \tilde{P}_{12} - n_{1,5}^0 - \zeta - \alpha - \beta \\ n_{1,4}^{12} = 4\tilde{P}_3 - 2\tilde{P}_5 + 2\tilde{P}_7 - \tilde{P}_8 - 2\tilde{P}_9 + \tilde{P}_{10} - \tilde{P}_{11} + \tilde{P}_{12} + n_{1,5}^0 - 2\eta + 2\zeta + \alpha + \beta \\ n_{2,9}^{12} = -2\tilde{P}_3 + \tilde{P}_4 + \tilde{P}_5 - \tilde{P}_7 + \tilde{P}_8 + \tilde{P}_9 - \tilde{P}_{10} - n_{1,5}^0 + \eta - \zeta \\ n_{1,5}^{12} = 2\tilde{P}_3 - \tilde{P}_4 - \tilde{P}_5 + \tilde{P}_7 - \tilde{P}_8 - \tilde{P}_9 + \tilde{P}_{10} + 2n_{1,5}^0 - \eta + \zeta \end{array} \right.$$

To recall the meaning of several symbols, η is the number of packing $\{(1, 3), (1, 4)\} \succ \{(2, 7)\}$, ζ is the number of packing $\{(1, 2), (3, 7)\} \succ \{(4, 9)\}$, α is the number of packing $\{(1, 2), (4, 9)\} \succ \{(5, 11)\}$ and β is the number of packing $\{(1, 3), (3, 8)\} \succ \{(4, 11)\}$.

6. Classification of baskets with $\chi = 1$

Assume that X is a minimal 3-fold of general type with $\chi(\mathcal{O}_X) = 1$. By Theorem 3.1, the canonical volume K_X^3 is bounded from below when $P_m \geq 2$ for some $m \leq 6$. It is thus sufficient to consider geometric formal basket under the following:

6.1. Assumption. Assume that B is a geometric basket on X and $P_m(X) \leq 1$ for $m \leq 6$.

In fact, one has the following geometric condition.

Lemma 6.2. *Let X be a minimal 3-fold of general type with $\chi(\mathcal{O}_X) = 1$. Then $P_{m+2} \geq P_m + P_2$ for all $m \geq 2$.*

Proof. By Reid's formula (4.1), we have

$$P_{m+2} - P_m - P_2 = (m^2 + m)K_X^3 - \chi(\mathcal{O}_X) + (l(m+2) - l(m) - l(2)).$$

By [18, Lemma 3.1], one sees $l(m+2) - l(m) - l(2) \geq 0$. Because $K_X^3 > 0$ and $\chi(\mathcal{O}_X) = 1$, we have $P_{m+2} - P_m - P_2 > -1$. The Lemma now follows. \square

We consider the formal basket

$$\mathbf{B} := \{B, \chi(\mathcal{O}_X), P_2\}.$$

Because B is geometric and $K_X^3(\mathbf{B}) = K_X^3 > 0$, the formal basket \mathbf{B} is admissible and positive. We may apply the argument in Section 5. Again because B is geometric, one sees $\tilde{P}_m = P_m(\mathbf{B}) = P_m(X)$ for all $m \geq 2$ and $\tilde{\chi} = \chi(\mathcal{O}_X) = 1$.

By Lemma 6.2, we see $P_4 \geq 2$ if $P_2 > 0$. Thus under Assumption 6.1, we have $P_2 = 0$. We can also get $P_{m+2} > 0$ whenever $P_m > 0$. Thus, in practice, we only need to study the following types: $P_2 = 0$ and

$$\begin{aligned} (P_3, P_4, P_5, P_6) = & (0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1), \\ & (0, 1, 0, 1), (0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 1, 1). \end{aligned}$$

If P_2, P_3, P_4, P_5 and P_6 are given, we are able to determine $\mathcal{B}^{(5)}(B)$. Our main task is to search all possible minimal (with regard to \succ) positive baskets dominated by $\mathcal{B}^{(5)}(B)$.

6.3. Notations and Conventions. Throughout this section, for a given basket B , we can consider the associated formal basket $\bar{B} := \{B, 1, 0\}$. We might abuse the notation of B and its associated formal basket \bar{B} in this section.

A basket with no further convenient packing is called *minimal*. A positive basket with no further convenient packing into a positive basket is called a *minimal positive basket*.

We let m_0 denote the minimal integer such that $P_{m_0} \geq 2$ from now on.

Now let us begin to classify all minimal positive geometric baskets.

6.4. The case I: $P_3 = P_4 = P_5 = P_6 = 0$.

Now we have $\sigma = 10, \tau = 4, \Delta^3 = 5, \Delta^4 = 14, \epsilon = 0, \sigma_5 = 0$ and $\epsilon_5 = 2$. The only possible initial basket is $\{5 \times (1, 2), 4 \times (1, 3), (1, 4)\}$. And $B^{(5)} = \{3 \times (1, 2), 2 \times (2, 5), 2 \times (1, 3), (1, 4)\}$ with $K^3 = \frac{1}{60}$.

We now classify baskets B with $\mathcal{B}^{(5)}(B)$ as above. This basket can only be obtained by successive convenient packings.

If we pack $\{(1, 2), (2, 5)\}$ to $\{(3, 7)\}$. Then we get

I-1. $\{(2, 4), (3, 7), (2, 5), (2, 6), (1, 4)\}$, $K^3 = \frac{1}{420}$, $m_0 = 18$,

which admits no further convenient packing into positive baskets. Hence it's minimal positive.

Thus it remains to consider baskets with $(1, 2)$ remained unpacked because otherwise it's dominated by the basket of Case I-1. So we consider the packing:

$\{(3, 6), (2, 5), (3, 8), (1, 3), (1, 4)\}$ with $K^3 = \frac{1}{120}$. This allows further packing to minimal positive ones:

I-2. $\{(3, 6), (2, 5), (4, 11), (1, 4)\}$, $K^3 = \frac{1}{220}$, $m_0 = 14$.

I-3. $\{(3, 6), (5, 13), (1, 3), (1, 4)\}$, $K^3 = \frac{1}{156}$, $m_0 = 12$.

It remains to consider the case that both $(1, 2)$ and $(2, 5)$ are remained unpacked. We can have the following packing which is indeed minimal positive:

I-4. $\{(3, 6), (4, 10), (1, 3), (2, 7)\}$, $K^3 = \frac{1}{210}$, $m_0 = 14$.

This gives a complete list of minimal positive baskets satisfying $P_2 = \dots = P_6 = 0$.

6.5. The case II: $P_3 = P_4 = P_5 = 0, P_6 = 1$.

Now we have $\sigma = 10, \tau = 4, \Delta^3 = 5, \Delta^4 = 14, \epsilon \leq 1$. If $\epsilon = 0$, then $\epsilon_5 = 1$ and if $\epsilon = 1$, then $\epsilon_5 = 0$. Thus the only possible initial baskets and $B^{(5)}$ are:

II-i. $B^{(0)} = \{5 \times (1, 2), 4 \times (1, 3), (1, 4)\} \succ B^{(5)} = \{4 \times (1, 2), (2, 5), 3 \times (1, 3), (1, 4)\}$, with $K^3(B^{(5)}) = \frac{1}{20}$.

II-ii. $B^{(0)} = \{5 \times (1, 2), 4 \times (1, 3), (1, 5)\} \succ B^{(5)} = \{5 \times (1, 2), 4 \times (1, 3), (1, 5)\}$, with $K^3(B^{(5)}) = \frac{1}{30}$.

In Case II-i, we first consider that all the basket $(1, 2)$ are packed with $(2, 5)$:

$\{(6, 13), 3 \times (1, 3), (1, 4)\}$. Further packing gives a minimal positive basket:

II-1. $\{(6, 13), (1, 3), (3, 10)\}$, $K^3 = \frac{1}{390}$, $m_0 = 9$.

We then consider the case that at least one basket $(1, 2)$ is remained unpacked. Then we reach:

II-2. $\{(1, 2), (5, 11), (4, 13)\}$, $K^3 = \frac{1}{286}$, $m_0 = 9$, which is minimal positive.

Notice that if $\{3 \times (1, 2), (3, 7), 3 \times (1, 3), (1, 4)\} \succ B$, then B dominate the basket in Case II-2. Thus it remains to consider the case that all basket $(1, 2)$ are remain unpacked and $(2, 5)$ must be packed with some $(1, 3)$. So we have minimal positive baskets:

II-3. $\{(4, 8), (3, 8), (3, 10)\}$, $K^3 = \frac{1}{40}$, $m_0 = 8$.

II-4. $\{(4, 8), (4, 11), (2, 7)\}$, $K^3 = \frac{2}{77}$, $m_0 = 8$.

II-5. $\{(4, 8), (5, 14), (1, 4)\}$, $K^3 = \frac{1}{28}$, $m_0 = 8$.

In case II-ii, $B^{(5)}$ admits no further convenient packing. Thus it's minimal positive.

II-6. $\{(5, 10), (4, 12), (1, 5)\}$, $K^3 = \frac{1}{30}$, $m_0 = 8$.

6.6. The case III: $P_3 = P_4 = 0, P_5 = 1, P_6 = 0$.

Now we have $\sigma = 10, \tau = 4, \Delta^3 = 5, \Delta^4 = 15$. Moreover, $P_7 \geq 1$, hence $\epsilon = 0, \sigma_5 = 0$ and $\epsilon_5 = 4$. Thus the only possible initial baskets and $B^{(5)}$ are:

$B^{(0)} = \{5 \times (1, 2), 5 \times (1, 3)\} \succ B^{(5)} = \{(1, 2), 4 \times (2, 5), (1, 3)\}$.

So the minimal positive baskets are:

III-1. $\{(9, 22), (1, 3)\}$, $K^3 = \frac{1}{66}$, $m_0 = 9$.

III-2. $\{(7, 17), (3, 8)\}$, $K^3 = \frac{1}{136}$, $m_0 = 10$.

III-3. $\{(5, 12), (5, 13)\}$, $K^3 = \frac{1}{156}$, $m_0 = 10$.

III-4. $\{(3, 7), (7, 18)\}$, $K^3 = \frac{1}{126}$, $m_0 = 10$.

III-5. $\{(1, 2), (9, 23)\}$, $K^3 = \frac{1}{46}$, $m_0 = 8$.

6.7. The case IV: $P_3 = P_4 = 0, P_5 = 1, P_6 = 1$.

Now we have $\sigma = 10, \tau = 4, \Delta^3 = 5, \Delta^4 = 15$. Moreover, the initial basket must have $n_{1,2}^0 = n_{1,3}^0 = 5$, hence $n_{1,r}^0 = 0$ for all $r \geq 4$. It follows that $\epsilon = 0, \sigma_5 = 0$ and $\epsilon_5 = 3$. Thus the only possible initial baskets and $B^{(5)}$ are:

$B^{(0)} = \{5 \times (1, 2), 5 \times (1, 3)\} \succ B^{(5)} = \{2 \times (1, 2), 3 \times (2, 5), 2 \times (1, 3)\}$.

So the minimal positive baskets are:

IV-1. $\{(8, 19), (2, 6)\} \succ \text{III} - 1$.

IV-2. $\{(6, 14), (4, 11)\} \succ \text{III} - 4$.

IV-3. $\{(4, 9), (6, 16)\} \succ \text{III} - 2$.

IV-4. $\{(2, 4), (8, 21)\} \succ \text{III} - 5$.

Hence the lower bound of K^3 can only be better.

6.8. The case V: $P_3 = 0, P_4 = 1, P_5 = 0, P_6 = 1$.

Now we have $\sigma = 10, \tau = 4, \Delta^3 = 6, \Delta^4 = 13$ and $\sigma_5 \leq \epsilon \leq 2$. The initial baskets could be :

V-i. $\{6 \times (1, 2), (1, 3), 3 \times (1, 4)\}$

V-ii. $\{6 \times (1, 2), (1, 3), 2 \times (1, 4), (1, 5)\}$

V-iii. $\{6 \times (1, 2), (1, 3), (1, 4), 2 \times (1, 5)\}$

V-iv. $\{6 \times (1, 2), (1, 3), 2 \times (1, 4), (1, r)\}$ with $r \geq 6$

The Case V-iii, V-iv are impossible since $K^3 \leq 0$. For Case V-i, we have $\epsilon_5 = 1$ and for case V-ii, we have $\epsilon_5 = 0$. Hence $B^{(5)}$ could be

V-i. $\{(5, 10), (2, 5), (3, 12)\}$

V-ii. $\{(6, 12), (1, 3), (2, 8), (1, 5)\}$

So the minimal positive baskets are:

V-1. $\{(7, 15), (3, 12)\}$, $K^3 = \frac{1}{60}$, $m_0 = 7$.

V-2. $\{(6, 12), (1, 3), (3, 13)\}$, $K^3 = \frac{1}{39}$, $m_0 = 8$.

V-3. $\{(6, 12), (3, 11), (1, 5)\}$, $K^3 = \frac{1}{55}$, $m_0 = 8$.

6.9. The case VI: $P_3 = 0, P_4 = P_5 = P_6 = 1$.

Now we have $\sigma = 10, \tau = 4, \Delta^3 = 6, \Delta^4 = 14$. Also $P_7 \geq 1$ and hence $\sigma_5 \leq \epsilon \leq 2$. The initial baskets could be :

VI-i. $\{6 \times (1, 2), 2 \times (1, 3), 2 \times (1, 4)\}$

VI-ii. $\{6 \times (1, 2), 2 \times (1, 3), (1, 4), (1, 5)\}$

VI-iii. $\{6 \times (1, 2), 2 \times (1, 3), 2 \times (1, 5)\}$

VI-iv. $\{6 \times (1, 2), 2 \times (1, 3), (1, 4), (1, r)\}$ with $r \geq 6$

Since there are only 2 baskets of $(1, 3)$, we have $\epsilon_5 = 3 - \sigma_5 \leq 2$. Hence $\sigma_5 > 0$ and $\epsilon > 0$. Therefore, case VI-i is impossible.

For Case VI-ii, $\epsilon_5 = 2$, hence

VI-ii. $B^{(5)} = \{4 \times (1, 2), 2 \times (2, 5), (1, 4), (1, 5)\}$.

Similarly, one can compute the minimal positive baskets:

IV-1. $\{(1, 2), (7, 16), (2, 9)\}, K^3 = \frac{1}{144}, m_0 = 7$.

IV-2. $\{(6, 13), (2, 5), (2, 9)\}, K^3 = \frac{8}{585}, m_0 = 7$.

IV-3. $\{(8, 18), (1, 4), (1, 5)\}, K^3 = \frac{1}{180}, m_0 = 7$.

For Case VI-iii, $\epsilon_5 = 1$, hence

VI-ii. $B^{(5)} = \{5 \times (1, 2), (2, 5), (1, 3), 2 \times (1, 5)\}$.

Then minimal positive baskets are:

IV-4. $\{(1, 2), (6, 13), (1, 3), (2, 10)\}, K^3 = \frac{1}{390}, m_0 = 8$.

IV-5. $\{(5, 10), (3, 8), (2, 10)\}, K^3 = \frac{1}{40}, m_0 = 8$.

For Case VI-iv, $\epsilon_5 = 2$, hence

VI-iv. $B^{(5)} = \{4 \times (1, 2), 2 \times (2, 5), (1, 4), (1, r)\}$ with $r \geq 6$.

Since $K^3(B^{(5)}) > 0$, we must have $r = 6$. Then the minimal positive basket is:

IV-6. $\{(3, 6), (3, 7), (2, 5), (1, 4), (1, 6)\}, K^3 = \frac{1}{420}, m_0 = 10$.

6.10. The case VII: $P_3 = 1, P_4 = 0, P_5 = P_6 = 1$.

Now we have $\sigma = 9, \tau = 3, \Delta^3 = 1, \Delta^4 = 9$. Moreover, $P_7 \geq 1$ and hence $\epsilon = 0$. It follows that $\sigma_5 = 0$ and $\epsilon_5 = 2$. The initial baskets is : $B^{(0)} = \{(1, 2), 7 \times (1, 3), (1, 4)\}$

Note that there is only one basket of type $(1, 2)$, while $\epsilon_5 = 2$ gives a contradiction since $n_{2,5}^5 = \epsilon_5 = 2$.

6.11. The case VIII: $P_3 = P_4 = P_5 = P_6 = 1$.

Now we have $\sigma = 9, \tau = 3, \Delta^3 = 2, \Delta^4 = 8$. Moreover, $P_7 \geq 1$ and then $\epsilon \leq 1$. If $\epsilon = 1$, then $\sigma_5 = 1$ and $\epsilon_5 = 1$. If $\epsilon = 0$, then $\sigma_5 = 0$ and $\epsilon_5 = 2$. The initial baskets and $B^{(5)}$ could be:

VIII-i. $B^{(0)} = \{2 \times (1, 2), 4 \times (1, 3), 3 \times (1, 4)\} \succ B^{(5)} = \{2 \times (2, 5), 2 \times (1, 3), 3 \times (1, 4)\}$, with $K^3(B^{(5)}) = \frac{1}{60}$.

VIII-ii. $B^{(0)} = \{2 \times (1, 2), 4 \times (1, 3), 2 \times (1, 4), (1, 5)\} \succ B^{(5)} = \{(1, 2), (2, 5), 3 \times (1, 3), 2 \times (1, 4), (1, 5)\}$, with $K^3(B^{(5)}) = 0$.

It's clear that Case VIII-ii is impossible since it is not positive.

For Case VIII-i, we first consider $\{(2, 5), (3, 8), (1, 3), (4, 12)\}$. It follows that it dominates two possible minimal positive baskets:

VIII-1. $\{(5, 13), (1, 3), (3, 12)\}$, $K^3 = \frac{1}{156}$, $m_0 = 7$.

VIII-2. $\{(2, 5), (4, 11), (3, 12)\}$, $K^3 = \frac{1}{220}$, $m_0 = 7$.

Now it remains to consider the case where $(2, 5)$ remains unpacked. We then consider $\{(4, 10), (1, 3), (2, 7), (2, 8)\}$ with $K^3 = \frac{1}{210}$. It allows the following minimal positive basket which is a one-step packing:

VIII-3. $\{(4, 10), (1, 3), (3, 11), (1, 4)\}$, $K^3 = \frac{1}{660}$, $m_0 = 7$.

For the canonical volume, we conclude the following:

Theorem 6.12. *Every minimal 3-fold X of general type with $\chi(\mathcal{O}_X) = 1$ has $K^3 \geq \frac{1}{420}$.*

Proof. By Theorem 3.1, we may assume that $P_m \leq 1$ for $m \leq 6$. Take the geometric formal basket $\mathbf{B}(X) = \{B, 1, P_2\}$ with $B = \tilde{\mathcal{B}}(X)$. Then by the definition of geometric basket, we have $K_X^3 = K^3(\mathbf{B})$. By our classification in above, we know $B \succ B_{\min}$ for a minimal positive basket B_{\min} of type I-1 through VIII-3. Because $\sigma(B_{\min}) = \sigma(B)$ and so $\sigma(B_{\min}) = 10 + 5P_2 - P_3$, we see that $\bar{B}_{\min} := \{B_{\min}, 1, P_2\}$ is a formal basket by definition. Furthermore we see $P_3(\bar{B}_{\min}) = P_3(X)$. Because Lemma 4.8(3) says $\sigma'(B) \geq \sigma'(B_{\min})$, we see that $K_X^3 = K^3(\mathbf{B}) \geq K^3(\bar{B}_{\min})$.

Now by above classification of minimal baskets, we have $K^3(\bar{B}_{\min}) \geq \frac{1}{420}$ unless B_{\min} is of Case VIII-3. Notice that in Case VIII-3, $P_7(X) = 2$ by direct computation. Thus by Table A in Section 3, we have $K_X^3 = K^3(\mathbf{B}) \geq \frac{5}{2408} > \frac{1}{660}$. So this means that the type VIII-3 minimal basket is not geometric. Notice that the type VIII-3 minimal basket is obtained by one-step packing from

$$B_{210} := \{(4, 10), (1, 3), (2, 7), (2, 8)\}$$

with $K^3(B_{210}) = \frac{1}{210} > \frac{1}{420}$. Furthermore, B_{210} is the only intermediate basket. Thus either B dominates a minimal basket of other types or B_{210} . So at any case we have seen $K_X^3 \geq \frac{1}{420}$. \square

The proof of the last theorem gives the following:

Corollary 6.13. *Let X be a minimal 3-fold of general type. Any geometric basket B on X either dominates a minimal basket of type different from VIII-3 or dominates the basket*

$$B_{210} := \{(4, 10), (1, 3), (2, 7), (2, 8)\}.$$

The lower bound of K^3 in Theorem 6.12 is optimal. Iano-Fletcher has already found the following example:

Example 6.14. ([16, p151, No.23]) The canonical hypersurface $X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$ has 7 terminal quotient singularities and the canonical volume $K_{X_{46}}^3 = \frac{1}{420}$. Because $p_g(X_{46}) = q(X_{46}) = h^2(\mathcal{O}_{X_{46}}) = 0$, one sees $\chi(\mathcal{O}_{X_{46}}) = 1$.

7. Classification of baskets with $\chi > 1$

In order to study the case with $\chi \geq 2$, we need to go further to develop the basket packing technique. We always consider the formal basket $\mathbf{B} = \{B, \chi(\mathcal{O}_X), P_2\}$ which is admissible and positive. Therefore we can apply our general theory in Section 5, just replacing $\tilde{\chi}$, and \tilde{P}_2 in Section 5 by $\chi(\mathcal{O}_X)$ and $P_2(X)$ respectively. All other symbols are also replaced accordingly.

By Theorem 3.1, we only need to study varieties under the following:

7.1. Assumption. Assume $P_m \leq 1$ for all $m \leq 12$.

Recalling equation (5.2), we have:

$$\epsilon_6 := -3P_2 - P_3 + P_4 + P_5 + P_6 - P_7 - \epsilon = 0$$

which is equivalent to

$$P_4 + P_5 + P_6 = 3P_2 + P_3 + P_7 + \epsilon. \quad (7.1)$$

Notice that, under Assumption 7.1 and if $P_2 = 1$, then by equation (7.1), $P_4 = P_5 = P_6 = 1$ and $P_3 = P_7 = \epsilon = 0$. But this is impossible since $P_2 = P_5 = 1$ implies $P_7 \geq 1$. Thus we may assume that $P_2 = 0$ in the following discussion.

Assumption 7.1 allows us to compute $B^{(12)}$. One can see that χ is bounded by P_m for $m \leq 12$ by the inequality (5.3). Hence it is possible to classify all possible basket $B^{(12)}$ under Assumption 7.1.

Note that the equality (5.3) will be as:

$$2P_5 + 3P_6 + P_8 + P_{10} + P_{12} \geq \chi + 10P_2 + 4P_3 + P_7 + P_{11} + P_{13} + R, \quad (7.2)$$

where

$$\begin{aligned} R &:= 14\sigma_5 - 12n_{1,5}^0 - 9n_{1,6}^0 - 8n_{1,7}^0 - 6n_{1,8}^0 - 4n_{1,9}^0 - 2n_{1,10}^0 - n_{1,11}^0 \\ &= 2n_{1,5}^0 + 5n_{1,6}^0 + 6n_{1,7}^0 + 8n_{1,8}^0 + 10n_{1,9}^0 + 12n_{1,10}^0 + 13n_{1,11}^0 + 14 \sum_{r \geq 12} n_{1,r}^0. \end{aligned}$$

and $\sigma_5 = \sum_{r \geq 5} n_{1,r}^0$.

With all these preparations, we are able to prove the following:

Theorem 7.2. *Let X be a minimal 3-fold of general type with $\chi(\mathcal{O}_X) \geq 2$. Then either $\chi(\mathcal{O}_X) \leq 6$ or $P_m \geq 2$ for some $m \leq 12$.*

Proof. If $P_m \leq 1$ for all $m \leq 12$, we have seen $P_2 = 0$. Then by (7.2), we get $8 \geq \chi = \chi(\mathcal{O}_X)$. If $\chi = 7$ or 8 , then $P_5 = P_6 = 1$. It follows that $P_{11} = 1$. Hence $8 \geq \chi + 1$ gives $\chi = 7$ and $P_8 = 1$ as well. Then $P_{13} = 1$. This leads to $8 \geq \chi + 2 = 9$, a contradiction. \square

7.3. Verifying Assumption 5.5. Assume $P_m \leq 1$ for all $m \leq 12$ and $\chi(\mathcal{O}_X) \geq 2$. Then we have seen $P_2 = 0$. We study $n_{1,r}^0$ when $r \geq 6$. If there exists a number $r \geq 6$ such that $n_{1,r}^0 \neq 0$, then $R \geq 5$ by the definition of R . Now (7.2) gives

$$8 \geq \chi + 5 \geq 7.$$

This implies that $P_5 = P_6 = 1$. Hence $P_{11} = 1$. Now (7.2) reads $5 + P_8 + P_{10} + P_{12} \geq 8 + P_7 + P_{13}$. One then has $P_8 = P_{10} = P_{12} = 1$ and $P_7 = P_{13} = 0$. This leads to a contradiction since $P_{13} \geq P_5 P_8 = 1$. So we conclude $n_{1,r}^0 = 0$ for all $r \geq 6$. In other words, Assumption 5.5 is satisfied.

This allows us to utilize those classifications in the last part of Section 5.

7.4. Classifying admissible baskets under extra conditions. We hope to classify all positive admissible baskets $\mathcal{B}^{(12)}(B)$ under Assumption 7.1 and $\chi(\mathcal{O}_X) \geq 2$. Note that, for all $0 < m, n \leq 12$, and $m + n \leq 13$,

$$P_{m+n} \geq P_m P_n \quad (7.3)$$

naturally holds since $P_m, P_n \leq 1$ for geometric formal basket. Our main result is Table B which is a complete list of all possibilities of $B^{(12)}$ and can be obtained by a simple computer program, or even by a direct, but time consuming calculation.

In fact, first we preset $P_m = 0, 1$ for $m = 3, \dots, 11$. Then $\epsilon_6 = 0$ gives the value of ϵ . So we know the value of $n_{1,5}^0$. By the inequality (5.2) we get the upper bound of χ since $P_{13} \geq 0$. Since $n_{1,4}^7 \geq 0$, we get the upper bound of η . Similarly $n_{2,9}^9 \geq 0$ gives the upper bound of ζ . Also $n_{4,9}^{11} \geq 0$ yields $\alpha \leq \zeta$. Finally $n_{3,8}^{11} \geq 0$ gives the upper bound of β . Now we set $P_{12} = 0, 1$. Then the inequality (5.3) again gives the upper bound of P_{13} , noting that $\chi \geq 2$. Clearly there are only finite many solutions. With inequality (7.3) imposed we can get only about 80 cases. An important relation to recall is $B^{(12)} \succ B$. So we see $K^3(B^{(12)}) \geq K^3(B) = K_X^3 > 0$. The final imposed property eventually outputs 63 cases which is exactly Table B.

All minimal positive baskets dominated by $B^{(12)}$ are also collected in Table B.

If one would like to take a direct calculation by hand, it is of course possible. Consider no.2 case in Table B as an example. Those formulae in Section 5 gives us enough information to compute $B^{(12)}$. Because $P_2 = 0$, $P_3 = \dots = P_7 = 0$, $P_8 = 1$ and $P_9 = P_{10} = P_{11} = 0$, (7.1) tells $\epsilon = 0$ and thus $\sigma_5 = 0$, which means $R = 0$. (7.2) gives $P_{12} + 1 \geq \chi + P_{13} \geq 2$. So $P_{12} = 1$, $\chi = 2$ and $P_{13} = 0$. Now the formula for ϵ_{10} gives $\epsilon_{10} = -\eta \geq 0$, which means $\eta = 0$. Similarly $n_{1,5}^9 = \zeta - 1 \geq 0$. On the other hand, $n_{3,7}^9 = 1 - \zeta \geq 0$. Thus $\zeta = 1$. Now $n_{4,9}^{11} = \zeta - \alpha \geq 0$ gives $\alpha \leq 1$. $n_{3,11}^{11} = 1 - \zeta - \alpha - \beta \geq 0$ gives $\alpha = \beta = 0$. Finally we get

$$\{n_{1,2}, n_{5,12}, \dots, n_{1,5}\} = \{4, 0, 1, 0, 0, 2, 1, 0, 3, 0, 0, 0, 2, 0, 0\}$$

That is $B^{(12)} = \{4 \times (1, 2), (4, 9), 2 \times (2, 5), (3, 8), 3 \times (1, 3), 2 \times (1, 4)\}$.

Table B.

$no.$	(P_3, \dots, P_{11})	P_{18}	P_{24}	m_0	χ	$B^{(12)} = (n_{1,2}, n_{5,11}, \dots, n_{1,5})$ or B_{min}	$K^3(B)$
1	(0, 0, 0, 0, 0, 0, 0, 1, 0)	4	8	14	2	(5, 0, 0, 1, 0, 3, 0, 0, 3, 0, 0, 1, 0, 0, 0)	$\frac{3}{770}$
2	(0, 0, 0, 0, 0, 1, 0, 0, 0)	3	7	15	2	(4, 0, 1, 0, 0, 2, 1, 0, 3, 0, 0, 0, 2, 0, 0)	$\frac{1}{360}$
2a		2	3	18		$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{1170}$
3	(0, 0, 0, 0, 0, 1, 0, 1, 0)	3	7	15	3	(6, 1, 0, 0, 0, 4, 1, 0, 4, 0, 1, 0, 2, 0, 0)	$\frac{23}{9240}$
3a		2	3	18		$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{13}{30030}$
4	(0, 0, 0, 0, 0, 1, 0, 1, 0)	4	9	14	3	(7, 0, 1, 0, 0, 4, 0, 1, 3, 0, 1, 0, 2, 0, 0)	$\frac{13}{3465}$
4a		1	2	14		$\{(4, 11), (2, 6), *\} \succ \{(6, 17), *\}$	$\frac{1}{5355}$
5	(0, 0, 0, 0, 0, 1, 0, 1, 0)	5	10	14	3	(7, 0, 1, 0, 0, 4, 1, 0, 4, 0, 0, 1, 1, 0, 0)	$\frac{17}{3960}$
5a		4	3	15		$\{(8, 20), (3, 8), *\} \succ \{(11, 28), *\}$	$\frac{1}{1386}$
5b		3	3	15		$\{(5, 13), (4, 15), *\}$	$\frac{1}{1170}$
6	(0, 0, 0, 1, 0, 0, 0, 1, 0)	3	6	14	3	(9, 0, 0, 2, 0, 1, 0, 1, 4, 0, 2, 0, 0, 0, 1)	$\frac{1}{462}$
7	(0, 0, 0, 1, 0, 0, 1, 0, 0)	3	5	14	2	(5, 0, 1, 1, 0, 0, 0, 0, 5, 0, 1, 0, 0, 0, 1)	$\frac{1}{630}$
7a		2	3	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{1680}$
8	(0, 0, 0, 1, 0, 0, 1, 1, 0)	3	5	14	3	(7, 1, 0, 1, 0, 2, 0, 0, 6, 0, 2, 0, 0, 0, 1)	$\frac{1}{770}$
9	(0, 0, 0, 1, 0, 1, 0, 0, 0)	2	2	14	3	(9, 0, 0, 2, 0, 0, 1, 1, 4, 0, 1, 0, 0, 1, 0)	$\frac{1}{5544}$
10	(0, 0, 0, 1, 0, 1, 0, 0, 0)	3	6	14	3	(8, 0, 1, 1, 0, 0, 2, 0, 5, 0, 1, 0, 1, 0, 1)	$\frac{1}{630}$
10a		2	4	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{1680}$
11	(0, 0, 0, 1, 0, 1, 0, 1, 0)	2	4	14	3	(9, 0, 0, 2, 0, 0, 1, 1, 3, 1, 0, 0, 1, 0, 1)	$\frac{3}{3080}$
11a		2	3	14		$\{(3, 8), (4, 11), *\} \succ \{(7, 19), *\}$	$\frac{1}{2660}$
12	(0, 0, 0, 1, 0, 1, 0, 1, 0)	5	11	14	3	(9, 0, 1, 0, 0, 1, 2, 0, 4, 0, 2, 0, 0, 0, 1)	$\frac{1}{252}$
12a		4	6	14		$\{(2, 5), (6, 16), *\} \succ \{(8, 21), *\}$	$\frac{1}{630}$
13	(0, 0, 0, 1, 0, 1, 0, 1, 0)	3	4	14	4	(12, 0, 0, 2, 0, 2, 0, 2, 4, 0, 2, 0, 0, 1, 0)	$\frac{1}{3465}$
14	(0, 0, 0, 1, 0, 1, 0, 1, 0)	3	6	14	4	(10, 1, 0, 1, 0, 2, 2, 0, 6, 0, 2, 0, 1, 0, 1)	$\frac{1}{770}$
15	(0, 0, 0, 1, 0, 1, 0, 1, 0)	4	8	14	4	(11, 0, 1, 1, 0, 2, 1, 1, 5, 0, 2, 0, 1, 0, 1)	$\frac{71}{27720}$
15a		2	4	14		$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{2520}$
15b		3	4	14		$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{23}{36036}$
15c		3	5	14		$\{(7, 16), (7, 19), *\}$	$\frac{31}{31920}$
16	(0, 0, 0, 1, 0, 1, 0, 1, 0)	5	9	14	4	(11, 0, 1, 1, 0, 2, 2, 0, 6, 0, 1, 1, 0, 0, 1)	$\frac{4}{13860}$
16a		4	3	14		$\{(4, 10), (3, 8), *\} \succ \{(7, 18), *\}$	$\frac{1}{3080}$
16b		4	4	14		$\{(2, 5), (6, 16), *\} \succ \{(8, 21), *\}$	$\frac{1}{1386}$
16c		3	3	14		$\{(7, 16), (5, 13), *\}$	$\frac{3}{16016}$
17	(0, 0, 0, 1, 0, 1, 0, 1, 1)	3	6	14	3	(9, 0, 0, 2, 0, 0, 0, 2, 3, 0, 1, 0, 1, 0, 1)	$\frac{3}{1540}$
18	(0, 0, 0, 1, 0, 1, 0, 1, 1)	4	7	14	3	(9, 0, 0, 2, 0, 0, 1, 1, 4, 0, 0, 1, 0, 0, 1)	$\frac{23}{9240}$
18a		2	3	14		$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{3080}$
18b		4	6	14		$\{(3, 8), (4, 11), *\} \succ \{(7, 19), *\}$	$\frac{83}{43890}$
19	(0, 0, 0, 1, 0, 1, 1, 0, 0)	3	3	14	3	(8, 0, 1, 1, 0, 1, 0, 1, 5, 0, 1, 0, 0, 1, 0)	$\frac{2}{3465}$
20	(0, 0, 0, 1, 0, 1, 1, 0, 0)	4	7	14	3	(7, 0, 2, 0, 0, 1, 1, 0, 6, 0, 1, 0, 1, 0, 1)	$\frac{1}{504}$
20a		3	3	18		$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{16380}$
21	(0, 0, 0, 1, 0, 1, 1, 1, 0)	4	8	14	2	(6, 0, 1, 0, 0, 0, 1, 0, 3, 1, 0, 0, 0, 0, 1)	$\frac{1}{360}$
21a		2	3	16		$\{(1, 3), (3, 10), *\} \succ \{(4, 13), *\}$	$\frac{1}{4680}$
22	(0, 0, 0, 1, 0, 1, 1, 1, 0)	2	3	18	3	(7, 1, 0, 1, 0, 1, 1, 0, 5, 1, 0, 0, 1, 0, 1)	$\frac{1}{9240}$
23	(0, 0, 0, 1, 0, 1, 1, 1, 0)	3	5	14	3	(8, 0, 1, 1, 0, 1, 0, 1, 4, 1, 0, 0, 1, 0, 1)	$\frac{19}{13860}$
23a		2	3	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{2640}$
24	(0, 0, 0, 1, 0, 1, 1, 1, 0)	3	3	14	4	(10, 1, 0, 1, 0, 3, 0, 1, 6, 0, 2, 0, 0, 1, 0)	$\frac{1}{3465}$
25	(0, 0, 0, 1, 0, 1, 1, 1, 0)	4	7	14	4	(9, 1, 1, 0, 0, 3, 1, 0, 7, 0, 2, 0, 1, 0, 1)	$\frac{4}{27720}$
25a		4	6	14		$\{(5, 11), (4, 9), *\} \succ \{(9, 20), *\}$	$\frac{41}{840}$
26	(0, 0, 0, 1, 0, 1, 1, 1, 0)	5	9	14	4	(10, 0, 2, 0, 0, 3, 0, 1, 6, 0, 2, 0, 1, 0, 1)	$\frac{41}{13860}$
26a		3	5	14		$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{1260}$
27	(0, 0, 0, 1, 0, 1, 1, 1, 0)	6	10	14	4	(10, 0, 2, 0, 0, 3, 1, 0, 7, 0, 1, 1, 0, 0, 1)	$\frac{97}{27720}$
27a		5	3	14		$\{(6, 15), (3, 8), *\} \succ \{(9, 23), *\}$	$\frac{19}{79695}$
27b		5	5	14		$\{(5, 13), (5, 18), *\}$	$\frac{1}{1170}$
28	(0, 0, 0, 1, 0, 1, 1, 1, 1)	4	8	14	2	(5, 1, 0, 0, 0, 0, 1, 0, 4, 0, 1, 0, 0, 0, 1)	$\frac{23}{9240}$
29	(0, 0, 0, 1, 0, 1, 1, 1, 1)	5	10	14	2	(6, 0, 1, 0, 0, 0, 0, 1, 3, 0, 1, 0, 0, 0, 1)	$\frac{13}{3465}$
29a		2	3	14		$\{(4, 11), (2, 6), *\} \succ \{(6, 17), *\}$	$\frac{1}{5355}$

$no.$	(P_3, \dots, P_{11})	P_{18}	P_{24}	m_0	χ	$(n_{1,2}, n_{4,9}, \dots, n_{1,5})$ or B_{min}	$K^3(B)$
30	(0, 0, 0, 1, 0, 1, 1, 1, 1)	3	5	14	3	(7, 1, 0, 1, 0, 1, 0, 1, 5, 0, 1, 0, 1, 0, 1)	$\frac{1}{924}$
31	(0, 0, 0, 1, 0, 1, 1, 1, 1, 1)	4	6	14	3	(7, 1, 0, 1, 0, 1, 1, 0, 6, 0, 0, 1, 0, 0, 1)	$\frac{1}{616}$
32	(0, 0, 0, 1, 0, 1, 1, 1, 1, 1)	5	8	14	3	(8, 0, 1, 1, 0, 1, 0, 1, 5, 0, 0, 1, 0, 0, 1)	$\frac{2}{693}$
32a		4	6	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{528}$
32b		2	2	14		$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{1386}$
33	(0, 0, 0, 1, 1, 0, 0, 1, 0)	2	4	14	2	(5, 0, 0, 2, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0)	$\frac{1}{840}$
33a		1	3	14		$\{(3, 10), (2, 7), *\} \succ \{(5, 17), *\}$	$\frac{1}{2856}$
34	(0, 0, 0, 1, 1, 0, 0, 1, 0)	4	8	14	3	(7, 0, 1, 1, 0, 2, 1, 0, 3, 0, 3, 0, 0, 0, 0)	$\frac{1}{360}$
34a		3	6	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{560}$
34b		3	4	14		$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{1170}$
35	(0, 0, 0, 1, 1, 0, 0, 1, 1)	3	6	14	2	(5, 0, 0, 2, 0, 0, 0, 1, 1, 0, 2, 0, 0, 0, 0)	$\frac{1}{462}$
36	(0, 0, 0, 1, 1, 0, 1, 1, 0)	3	5	14	2	(4, 0, 1, 1, 0, 1, 0, 0, 2, 1, 1, 0, 0, 0, 0)	$\frac{1}{630}$
36a		2	3	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{1680}$
36b		2	4	14		$\{(3, 10), (2, 7), *\} \succ \{(5, 17), *\}$	$\frac{4}{5355}$
37	(0, 0, 0, 1, 1, 0, 1, 1, 0)	5	9	14	3	(6, 0, 2, 0, 0, 3, 0, 0, 4, 0, 3, 0, 0, 0, 0)	$\frac{1}{315}$
38	(0, 0, 0, 1, 1, 0, 1, 1, 1)	3	5	14	2	(3, 1, 0, 1, 0, 1, 0, 0, 3, 0, 2, 0, 0, 0, 0)	$\frac{1}{770}$
39	(0, 0, 0, 1, 1, 1, 0, 1, 0)	3	6	14	3	(7, 0, 1, 1, 0, 1, 2, 0, 2, 1, 1, 0, 1, 0, 0)	$\frac{1}{630}$
39a		2	4	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{1680}$
39b		2	5	14		$\{(3, 10), (2, 7), *\} \succ \{(5, 17), *\}$	$\frac{4}{5355}$
40	(0, 0, 0, 1, 1, 1, 0, 1, 0)	5	10	14	4	(9, 0, 2, 0, 0, 3, 2, 0, 4, 0, 3, 0, 1, 0, 0)	$\frac{1}{315}$
40a		4	4	14		$\{(4, 10), (3, 8), *\} \succ \{(7, 18), *\}$	$\frac{1}{2520}$
40b		4	5	14		$\{(2, 5), (6, 16), *\} \succ \{(8, 21), *\}$	$\frac{1}{1260}$
41	(0, 0, 0, 1, 1, 1, 0, 1, 1)	5	11	13	2	(5, 0, 1, 0, 0, 0, 2, 0, 1, 0, 2, 0, 0, 0, 0)	$\frac{1}{252}$
42	(0, 0, 0, 1, 1, 1, 0, 1, 1)	3	6	14	3	(6, 1, 0, 1, 0, 1, 2, 0, 3, 0, 2, 0, 1, 0, 0)	$\frac{1}{770}$
43	(0, 0, 0, 1, 1, 1, 0, 1, 1)	4	8	14	3	(7, 0, 1, 1, 0, 1, 1, 1, 2, 0, 2, 0, 1, 0, 0)	$\frac{1}{71}$
43a		2	4	14		$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{27720}$
43b		3	4	14		$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{2520}$
43c		3	5	14		$\{(7, 16), (7, 19), *\}$	$\frac{23}{36036}$
44	(0, 0, 0, 1, 1, 1, 0, 1, 1)	5	9	14	3	(7, 0, 1, 1, 0, 1, 2, 0, 3, 0, 1, 1, 0, 0, 0)	$\frac{31}{31920}$
44a		4	4	14		$\{(2, 5), (6, 16), *\} \succ \{(8, 21), *\}$	$\frac{43}{13860}$
44b		3	3	14		$\{(7, 16), (5, 13), *\}$	$\frac{3}{1386}$
44c		4	6	14		$\{(7, 16), (5, 18), *\}$	$\frac{1}{16016}$
44d		4	4	14		$\{(5, 13), (5, 18), *\}$	$\frac{1}{720}$
45	(0, 0, 0, 1, 1, 1, 1, 0, 1)	4	7	14	2	(3, 0, 2, 0, 0, 0, 1, 0, 3, 0, 1, 0, 1, 0, 0)	$\frac{1}{2184}$
46	(0, 0, 0, 1, 1, 1, 1, 1, 0)	4	7	14	3	(6, 0, 2, 0, 0, 2, 1, 0, 3, 1, 1, 0, 1, 0, 0)	$\frac{1}{504}$
46a		3	3	16		$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{504}$
46b		3	6	14		$\{(3, 10), (2, 7), *\} \succ \{(5, 17), *\}$	$\frac{1}{16380}$
47	(0, 0, 0, 1, 1, 1, 1, 1, 1)	2	3	16	2	(3, 1, 0, 1, 0, 0, 1, 0, 2, 1, 0, 0, 1, 0, 0)	$\frac{1}{6120}$
48	(0, 0, 0, 1, 1, 1, 1, 1, 1)	3	5	14	2	(4, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0)	$\frac{1}{9240}$
48a		2	3	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{19}{13860}$
49	(0, 0, 0, 1, 1, 1, 1, 1, 1)	4	7	14	3	(5, 1, 1, 0, 0, 2, 1, 0, 4, 0, 2, 0, 1, 0, 0)	$\frac{1}{2640}$
49a		4	6	14		$\{(5, 11), (4, 9), *\} \succ \{(9, 20), *\}$	$\frac{47}{27720}$
50	(0, 0, 0, 1, 1, 1, 1, 1, 1)	5	9	14	3	(6, 0, 2, 0, 0, 2, 0, 1, 3, 0, 2, 0, 1, 0, 0)	$\frac{1}{840}$
50a		3	5	14		$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{41}{13860}$
51	(0, 0, 0, 1, 1, 1, 1, 1, 1)	6	10	14	3	(6, 0, 2, 0, 0, 2, 1, 0, 4, 0, 1, 1, 0, 0, 0)	$\frac{1}{1260}$
51a		5	4	14		$\{(4, 10), (3, 8), *\} \succ \{(7, 18), *\}$	$\frac{97}{27720}$
51b		5	5	14		$\{(5, 13), (5, 18), *\}$	$\frac{1}{1386}$
52	(0, 0, 1, 0, 0, 1, 0, 1, 0)	3	7	14	2	(4, 0, 0, 1, 0, 2, 2, 0, 2, 0, 0, 0, 0, 0, 1)	$\frac{1}{1170}$
52a		2	3	18		$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{420}$
53	(0, 0, 1, 0, 0, 1, 1, 1, 0)	4	8	14	2	(3, 0, 1, 0, 0, 3, 1, 0, 3, 0, 0, 0, 0, 0, 1)	$\frac{1}{2184}$
53a		3	4	15		$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{360}$
54	(0, 0, 1, 0, 1, 0, 0, 1, 0)	2	4	14	2	(2, 0, 0, 2, 0, 3, 1, 0, 1, 0, 1, 0, 0, 0, 0)	$\frac{1}{1170}$
55	(0, 0, 1, 0, 1, 0, 0, 1, 0)	2	2	14	3	(4, 0, 0, 3, 0, 4, 1, 0, 3, 0, 0, 1, 0, 0, 0)	$\frac{1}{840}$
56	(0, 0, 1, 0, 1, 0, 1, 1, 0)	3	5	14	2	(1, 0, 1, 1, 0, 4, 0, 0, 2, 0, 1, 0, 0, 0, 0)	$\frac{1}{3080}$
56a		2	3	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{630}$

$no.$	(P_3, \dots, P_{11})	P_{18}	P_{24}	m_0	χ	$(n_{1,2}, n_{4,9}, \dots, n_{1,5})$ or B_{min}	$K^3(B)$
57	(0, 0, 1, 0, 1, 0, 1, 1, 0)	3	3	14	3	(3, 0, 1, 2, 0, 5, 0, 0, 4, 0, 0, 1, 0, 0, 0)	$\frac{1}{1386}$
58	(0, 0, 1, 0, 1, 1, 0, 1, 0)	3	6	14	3	(4, 0, 1, 1, 0, 4, 2, 0, 2, 0, 1, 0, 1, 0, 0)	$\frac{1}{630}$
58a		2	4	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{1680}$
59	(0, 0, 1, 0, 1, 1, 0, 1, 1)	2	4	14	2	(2, 0, 0, 2, 0, 2, 1, 1, 0, 0, 0, 0, 1, 0, 0)	$\frac{3}{3080}$
59a		2	3	14		$\{(3, 8), (4, 11), *\} \succ \{(7, 19), *\}$	$\frac{1}{2660}$
60	(0, 0, 1, 0, 1, 1, 1, 1, 0)	4	7	14	3	(3, 0, 2, 0, 0, 5, 1, 0, 3, 0, 1, 0, 1, 0, 0)	$\frac{1}{504}$
60a		3	3	15		$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{16380}$
61	(0, 0, 1, 0, 1, 1, 1, 1, 1)	2	3	15	2	(0, 1, 0, 1, 0, 3, 1, 0, 2, 0, 0, 0, 1, 0, 0)	$\frac{1}{9240}$
62	(0, 0, 1, 0, 1, 1, 1, 1, 1)	3	5	14	2	(1, 0, 1, 1, 0, 3, 0, 1, 1, 0, 0, 0, 1, 0, 0)	$\frac{19}{13860}$
62a		2	3	14		$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{2640}$
63	(0, 0, 1, 1, 1, 1, 1, 1, 1)	3	4	14	3	(5, 0, 1, 2, 0, 1, 1, 1, 3, 0, 1, 0, 0, 0, 1)	$\frac{1}{5544}$

We see that there is only one packing $\{(2, 5), (3, 8)\} \succ \{(5, 13)\}$ to get a minimal positive basket $\{4 \times (1, 2), (4, 9), (2, 5), (5, 13), 3 \times (1, 3), 2 \times (1, 4)\}$. We simply write this as $\{(5, 13), *\}$ in Table B. It is now easy to calculate K^3 for both $B^{(12)}$ and the minimal positive basket $\{(5, 13), *\}$. Finally we can directly calculate P_m . We use m_0 to denote the minimal integer with $P_{m_0} \geq 2$. For the need of our argument, we also display the value of $P_{18} = P_{18}(X)$ and $P_{24} = P_{24}(X)$ in Table B. So theoretically we can finish our classification by detailed computations. We omit more details because all calculations are similar.

One will see that many positive minimal baskets in Table B are not geometric. By Theorem 3.1 we know some effective lower bounds of K_X^3 . On the other hand, if we know a concrete m_0 and the volume of a minimal positive basket is smaller than the lower bound predicted in Theorem 3.1, then such a minimal positive basket would not be a geometric one.

Looking through Table B, we have:

Claim C. *Each minimal positive basket in Table B, of cases 4a, 9, 16a, 16c, 18a, 20a, 21a, 22, 24, 27a, 29a, 33a, 44b, 46a, 47, 52a, 55, 60a, 61, 63 is not geometric.*

Proof. 1). If $P_{14} \geq 2$, then $K^3 \geq \frac{11}{37800} > \frac{1}{3437}$ by Theorem 3.1. So the cases 4a, 9, 16c, 24, 27a, 29a, 44b, 63 are not geometric.

2). If $P_{15} \geq 2$, then $K^3 \geq \frac{11}{46080} > \frac{1}{4190}$ by Theorem 3.1, hence case 60a, 61 are not geometric.

3). If $P_{16} \geq 2$, then $K^3 \geq \frac{11}{55488} > \frac{1}{5045}$ by Theorem 3.1, hence the cases 46a, 47 are not geometric.

4). If $P_{18} \geq 2$, then $K^3 \geq \frac{11}{77976} > \frac{1}{7089}$ by Theorem 3.1. Thus the cases 20a, 22 are not geometric.

5). The case 33a has $P_6 = 1, P_{16} = 2$ but $P_{22} = 1$, a contradiction. So case 33a is not geometric.

6). The cases 16a, 18a, 21a, 52a, 55 have $P_{17} = 0$. In case 21a, $P_8 = P_9 = 1$, a contradiction. And in case 52a, 55, $P_5 = P_{12} = 1$, a contradiction. For case 18a, $P_6 = P_{11} = 1$, again a contradiction.

Finally in case 16a, computation shows that $P_{19} = -1$. Hence each of these cases is not geometric. \square

Theorem 7.5. *The canonical volume $K^3 \geq \frac{1}{2660}$ for all projective 3-folds of general type.*

Proof. It suffices to study minimal models. Let X be a minimal 3-fold X of general type. If $P_m \geq 2$ for some $m \leq 12$, then $K^3 \geq \frac{11}{24336} > \frac{1}{2213}$ by Theorem 3.1. Also notice that if $\chi(\mathcal{O}_X) \leq 0$, then $K^3 \geq \frac{1}{30}$ by [5, Theorem 1.1]. When $\chi = 1$, we have seen that $K^3 \geq \frac{1}{420}$ by Theorem 6.12. It remains to treat the case that $\chi(\mathcal{O}_X) > 1$ and $P_m \leq 1$ for $m \leq 12$.

Recall that we can study a geometric basket B on X and the corresponding formal basket $\mathbf{B} = \{B, \chi, P_2\}$. We have given Table B for each possible minimal positive basket B_{min} dominated by B . Clearly we have $K_X^3 = K^3(B) \geq K^3(B_{min})$ since $B \succ B_{min}$ for some B_{min} in Table B.

If B_{min} is geometric, then by searching in Table B and eliminate those non-geometric ones, we get $K^3 \geq \frac{1}{2660}$, which happens in cases 11a and 59a.

We still need to treat the case that B_{min} is non-geometric. Notice that if $B^{(12)}$ is minimal and non-geometric. Then $B^{(12)} \succ B \succ B_{min}$ gives $B^{(12)} = B$, which is non-geometric. This is a contradiction.

Hence it remains to consider those intermediate baskets between $B^{(12)}$ and B_{min} under the situation that $B^{(12)} \neq B_{min}$ and B_{min} is non-geometric.

Take case 4a for example. It's obtained by 2-steps packing

$$\{(2, 6), (4, 11)\} \succ \{(1, 3), (5, 14)\} \succ \{(6, 17)\}.$$

It doesn't really matter whether the intermediate basket B_{mid} is geometric or not. The intermediate basket has $K^3(B_{mid}) = \frac{1}{630} \geq \frac{1}{2660}$. Thus $K_X^3 > \frac{1}{2660}$.

One can see that the computation for case 29a is exactly the same.

Take case 33a for another example. It's obtained by 1-step packing $\{(3, 10), (2, 7)\} \succ \{(5, 17)\}$. Hence there is no intermediate baskets. So $B = B^{(12)}$ and $K_X^3 = K^3(B^{(12)}) = \frac{1}{840}$. Similar argument works for cases 18a, 20a, 21a, 46a, 52a, 60a.

Indeed the remaining cases are 16a, 16c, 27a, 44b. In case 44b, there are two intermediate baskets which dominates case 44c or 44d respectively. Thus in particular $K_X^3 > \frac{1}{2184}$. In case 27a, it's obtained from case 54 by 3-steps packing $\{3 \times (2, 5), (5, 8), *\} \succ \{2 \times (2, 5), (5, 13), *\} \succ \{(2, 5), (7, 18), *\} \succ \{(9, 23), *\}$. Every geometric basket dominating the basket in case 54a must dominate the basket $\{(2, 5), (7, 18), *\}$ with $K^3 = \frac{1}{1386}$. Finally, we consider cases 16a, 16c. The basket for case 16 is of the form $\{(4, 9), (3, 7), (2, 5), (2, 5), (3, 8), (3, 8), *\}$. If we take 1-step packing $\{(4, 9), (3, 7), (2, 5), (5, 13), (3, 8), *\}$, then this

is a common intermediate basket B_{mid} between $B^{(12)}$ and 16a or 16c. It has $K_{B_{mid}}^3 = \frac{85}{72072} > \frac{1}{848}$. The only remaining intermediate basket is $\{(7, 16), (2, 5), (2, 5), (3, 8), (3, 8), *\}$, which has the volume $\frac{13}{6160} > \frac{1}{474}$.

Thus for all non-geometric basket cases, we still have $K_X^3 > \frac{1}{2660}$. We have proved the theorem. \square

8. Birationality

With the technique of studying pluricanonical maps and the classification of baskets, we are able to study various explicit birational geometry including plurigenera and the pluricanonical birationality.

We will need the following:

Lemma 8.1. *Consider two formal baskets $\mathbf{B}_i = \{B_i, \tilde{\chi}, \tilde{P}_2\}$ for $i = 1, 2$. Assume $B_1 \succ B_2$ with $\mathcal{B}^{(0)}(B_1) = \mathcal{B}^{(0)}(B_2)$. Then we have $P_m(\mathbf{B}_1) \geq P_m(\mathbf{B}_2)$ for all $m \geq 2$.*

Proof. Since $\mathcal{B}^{(0)}(B_1) = \mathcal{B}^{(0)}(B_2)$, it follows by definition that $\sigma(B_1) = \sigma(B_2)$ and $\Delta^j(\mathbf{B}_1) = \Delta^j(\mathbf{B}_2)$ for $j = 3, 4$. Also we know $K^3(\mathbf{B}_1) - \sigma'(B_1) = K^3(\mathbf{B}_2) - \sigma'(B_2)$. By Lemma 4.8, we get $\Delta^m(\mathbf{B}_1) \geq \Delta^m(\mathbf{B}_2)$ for all $m \geq 5$.

Therefore by the inductive definition of $P_m(\mathbf{B}_i)$, we see $P_m(\mathbf{B}_1) \geq P_m(\mathbf{B}_2)$ for all $m \geq 2$. \square

Let us recall some known relevant results as follows. On irregular 3-folds there is already a practical result. The following theorem was proved by the first author and C. D. Hacon.

Theorem 8.2. [4] *Let X be a minimal projective 3-fold of general type with $q(X) := h^1(\mathcal{O}_X) > 0$. Then $P_m > 0$ for all $m \geq 2$ and φ_m is birational for all $m \geq 7$.*

Therefore, we do not need to worry about irregular 3-folds in the following discussion. The following result is due to Kollár.

Theorem 8.3. [21, Corollary 4.8] *Let X be a minimal projective 3-fold of general type with $P_{m_0} \geq 2$. Then φ_{11m_0+5} is birational onto its image.*

Kollár's result was ever improved by the second author:

Theorem 8.4. [9, Theorem 0.1] *Let X be a minimal projective 3-fold of general type with $P_{m_0} \geq 2$. Then φ_m is birational onto its image for all $m \geq 5m_0 + 6$.*

When $\chi(\mathcal{O}_X) < 0$, Reid's formula (4.1) says $P_2 \geq 4$ and $P_m > 0$ for all $m \geq 2$. So φ_m is birational for all $m \geq 16$ by Theorem 8.4.

When $\chi(\mathcal{O}_X) = 0$, since one can verify $l_Q(3) \geq l_Q(2)$ for any basket Q , Reid's formula (4.1) says: $P_3(X) > P_2(X) > 0$. Moreover, $P_{m+1} \geq P_m$ for all $m \geq 2$. Now $P_3(X) \geq 2$, so φ_m is birational for all $m \geq 21$ by Theorem 8.4.

To make a summary, we have the following result when $\chi \leq 0$.

Theorem 8.5. *let X be a minimal projective 3-fold of general type with $\chi(\mathcal{O}_X) \leq 0$. Then*

- (1) $P_m > 0$ for all $m \geq 2$;
- (2) $P_m \geq 2$ for all $m \geq 3$;
- (3) φ_m is birational for all $m \geq 21$.

Remark 8.6. Under the same condition as that of Theorem 8.5, K. Zuo and the second author [12] have actually proved that φ_m is birational for all $m \geq 14$ (optimal). Since the mentioned paper is not published yet, we list here Theorem 8.5 to make this paper more self-contained.

Now we recall Fletcher's interesting result.

Theorem 8.7. ([18]) *Let X be a minimal projective 3-fold of general type with $\chi = 1$. Then $P_{12} \geq 1$, $P_{24} \geq 2$.*

We are able to prove a more general result for all 3-folds of general type.

Theorem 8.8. *Let X be a minimal projective 3-fold of general type. Then $P_{12} \geq 1$.*

Proof. It suffices to prove this when $\chi \geq 2$ by Theorem 8.5 and 8.7. We assume $P_{12} = 0$ and will deduce a contradiction. It's clear that $P_2 = P_3 = P_4 = P_6 = 0$. We consider the geometric formal basket $\mathbf{B}(X) = \{B, \chi, P_2, P_3\}$.

Step 1. If $P_5 = 0$, then the equality (5.2) for ϵ_6 gives $P_7 = \epsilon = 0$. This also means $\sigma_5 = 0$. Hence Assumption 5.5 is satisfied. Now since $\epsilon_7 \geq \eta$ and $\epsilon_{12} \geq 0$, one gets

$$\chi \geq P_8 + \eta \geq \chi + P_{13}.$$

It follows that $\chi = P_8 + \eta$, $\epsilon_7 = \eta$ and $n_{3,7}^7 = 0$. Since $n_{3,7}^9 = -\zeta$, we have $\zeta = 0$. Now $n_{4,9}^{11} = \zeta - \alpha \geq 0$ gives $\alpha = 0$.

Hence since $n_{1,5}^0 = 0$ and so $n_{2,9}^9 = -n_{1,5}^9 = 0$, we have $n_{2,9}^9 = 0$ and $\epsilon_9 = n_{2,9}^9 + \zeta = 0$ which gives $P_{10} = P_8 + P_9 + \eta$.

Now $n_{3,8}^{12} + n_{2,7}^{12} \geq 0$ gives $\eta \geq \chi + 3P_9 \geq \eta + P_8 + 3P_9$. Hence $P_8 = P_9 = 0$, and also $P_{10} = \eta = \chi$. However, $n_{3,8}^{12} + n_{1,4}^{12} = P_{10} - 2\eta - P_{11} = -\chi - P_{11} < 0$, which is a contradiction.

Step 2. If $P_5 > 0$, then we have $P_7 = 0$. First of all, (5.2) gives $P_5 = \epsilon := n_{1,5}^0 + 2 \sum_{r \geq 6} n_{1,r}^0$. By definition of η , we see $\epsilon_7 \geq \eta$. Because $\epsilon_{12} \geq 0$, we get $\chi \geq P_8 + \eta + (2\sigma_5 - n_{1,5}^0 - n_{1,6}^0)$ and $2P_5 + P_8 + \eta \geq \chi + P_{13} + (8\sigma_5 - 7n_{1,5}^0 - 5n_{1,6}^0 - 5n_{1,7}^0 - \dots - n_{1,11}^0)$. Combine these two inequalities, we get

$$2\epsilon + P_8 + \eta = 2P_5 + P_8 + \eta \geq P_8 + P_{13} + \eta + R', \quad (8.1)$$

where $R' = 2n_{1,5}^0 + 4n_{1,6}^0 + 5n_{1,7}^0 + \dots + 9n_{1,11}^0 + 10 \sum_{r \geq 12} n_{1,r}^0 \geq 0$. Notice that $R' \geq 2\epsilon = 2P_5$. It follows that $P_{13} = 0$ and $n_{1,r}^0 = 0$ for all $r \geq 7$. Note also that $P_{13} = 0$ and $P_5 > 0$ implies $P_8 = 0$.

Step 3. We summarize that $\sigma_5 = n_{1,5}^0 + n_{1,6}^0$ and $P_5 = n_{1,5}^0 + 2n_{1,6}^0$. Now $\epsilon_7 \geq \eta$ and $\epsilon_{12} \geq 0$ reads

$$\chi \geq P_8 + \eta + n_{1,5}^0 + n_{1,6}^0 \geq \chi.$$

It follows that $\chi = P_8 + \eta + \sigma_5$. We then look at $n_{1,4}^7 = \chi - P_5 - \sigma_5 - \eta$. $n_{1,4}^7 \geq 0$ implies that $P_8 \geq P_5 > 0$, a contradiction. \square

Theorem 8.9. *Let X be a minimal projective 3-fold of general type. Then $P_{24} \geq 2$.*

Proof. It suffices to prove the theorem for $\chi \geq 2$ by Theorem 8.5 and 8.7. Also we only have to study the situation with $q(X) = 0$ by [4].

Suppose that $P_m \leq 1$ for all $m \leq 12$. Then by our classification of baskets in Table B, we have $P_{24}(B_{\min}) \geq 2$ for all minimal positive baskets. Thus by Lemma 8.1, we have $P_{24} = P_{24}(\mathbf{B}) \geq P_{24}(B_{\min}) \geq 2$, where \mathbf{B} is a geometric formal basket on X .

It remains to consider the case that $\chi \geq 2$ and $P_m \geq 2$ for some $m \leq 12$. Clearly, we only need to consider the cases with $m = 5, 7, 9, 10, 11$.

In what follows, we assume $P_{24} = 1$ and will deduce a contradiction. By Theorem 8.8, we may and do assume that $P_{12} = 1$. We will frequently use the following easy observation frequently:

$$P_{m+n} \geq P_m P_n \text{ if either } P_m \text{ or } P_n \leq 1 \quad (8.2).$$

Step 1. Suppose $P_{m_0} \geq 2$ for certain $m_0 \leq 10$. We can study φ_{m_0} . As we know that, because $\chi(\mathcal{O}_X) > 1$, φ_{m_0} is of type III, II, and I_p . By Proposition 3.5, we see that $P_m \geq 2$ for all $m \geq 2m_0 + 3 \geq 23$. In particular, $P_{24} \geq 2$.

Step 2. Suppose $P_{11} \geq 2$ and $P_m \leq 1$ for all $m \leq 10$. Clearly, $P_2 = 0$ since $P_{24} \geq P_2 P_{2 \times 11}$. First of all, $\epsilon_{10} \geq 0$ gives

$$2P_6 + P_{10} \geq P_{11} + P_3 + \eta + (n_{1,5}^0 + 2n_{1,6}^0 + 3n_{1,7}^0 + 4n_{1,8}^0 + 5n_{1,9}^0 + 6 \sum_{r \geq 10} n_{1,r}^0) \geq 2. \quad (8.3)$$

If $P_6 = 0$ then $P_{10} \geq 2$ which is absurd. Thus we may assume that $P_6 = 1$. Also $P_7 = 0$ since $P_{24} \geq P_6 P_7 P_{11}$.

If $P_3 = 1$, then ϵ_{10} gives $2 + P_{10} \geq P_{11} + P_3 \geq 3$. Hence $P_{10} = 1$. But then $P_{24} \geq P_{11} P_{13} \geq 2$, a contradiction. Thus we may assume $P_3 = 0$.

Now we look at (8.3) again. Since $3 \geq 2P_6 + P_{10}$, together with $P_{11} \geq 2$, we have $n_{1,r}^0 = 0$ for all $r \geq 6$ and $n_{1,5}^0 \leq 1$. Hence $\epsilon := n_{1,5}^0 + \sum_{r \geq 6} 2n_{1,r}^0 \leq 1$. Recall that $\epsilon_6 = 0$, which gives $1 + P_4 + P_5 = P_3 + P_7 + \epsilon \leq 1$. It follows that $P_4 = P_5 = 0$ and $\epsilon = 1$.

We make a summary: $P_2 = P_3 = P_4 = P_5 = P_7 = 0$, $P_6 = 1$.

Now ϵ_{10} gives

$$3 \geq 2 + P_{10} \geq P_{11} + 1 + \eta \geq 3.$$

We thus have $P_{10} = 1$, $P_{11} = 2$ and $\eta = 0$.

We then look at ϵ_{12} which says:

$$1 + P_8 + P_{12} \geq \chi + P_{13} + 1 \geq 3.$$

Thus $\chi = 2$, $P_8 = P_{12} = 1$ and $P_{13} = 0$. Also $\epsilon_9 \geq 0$ gives $P_9 = 1$.

With all above information, we see $n_{2,5}^8 = -1 < 0$ which is a contradiction.

This completes the proof. \square

Theorem 8.10. *Let X be a minimal projective 3-fold of general type. Then there exists an integer $m_0 \leq 18$ such that $P_{m_0} \geq 2$.*

Proof. If $\chi \leq 0$, then this is clear by Theorem 8.5. If $\chi = 1$, then either $P_m \geq 2$ for some $m \leq 6$ or the geometric basket $\mathbf{B} \succ B_{\min}$ for some B_{\min} in the list of Section 6. Note that for all baskets in the list of Section 6, we have $P_{m_0}(B_{\min}) \geq 2$ for some $m_0 \leq 18$. By Lemma 8.1, we have $P_{m_0}(X) = P_{m_0}(\mathbf{B}) \geq P_{m_0}(B_{\min}) \geq 2$ for some $m_0 \leq 18$.

If $\chi > 1$, then either $P_m \geq 2$ for some $m \leq 12$ or $B \succ B_{\min}$ for some B_{\min} in Table, Section 7. Similarly, for all minimal basket in table B, we can verify that there is some $m_0 \leq 18$ with $P_{m_0}(B_{\min}) \geq 2$. Hence again by Lemma 8.1, we have $P_{m_0}(X) \geq 2$ for some $m_0 \leq 18$. \square

Theorem 8.11. *For all minimal projective folds of general type, $m_1 \leq 27$. That is $P_m > 0$ for all $m \geq 27$.*

Proof. If $\chi < 1$, this is clear by Theorem 8.5.

If $\chi = 1$, then either $P_{m_0} \geq 2$ for some $m_0 \leq 6$ or it's classified in Section 6. By Proposition 3.5, $m_1 \leq 3m_0 + 4 \leq 22$ if $P_{m_0} \geq 2$ for some $m_0 \leq 6$. For those minimal baskets B_{\min} classified in Section 6, direct computation shows that $P_m(B_{\min}) > 0$ for $m \geq 14$. Thus $P_m(X) = P_m(\mathbf{B}) \geq 0$ by lemma 8.1 for all $m \geq 14$.

If $\chi \geq 2$, then either $P_{m_0} \geq 2$ for some $m_0 \leq 12$ or the minimal basket B_{\min} of the geometric basket B is classified in Section 7. Notice that when $P_{m_0} \geq 2$ for some $m_0 \leq 12$, the induced map φ_{m_0} can not be of type I_n as we pointed out in Remark 3.6. Therefore, by Proposition 3.5, $m_1 \leq 2m_0 + 3 \leq 27$ in this situation. It remains to consider those baskets classified in Section 7. A direct but tedious computation shows that $P_m(B_{\min}) > 0$ for $m \geq 24$. Thus $P_m(X) = P_m(\mathbf{B}) > 0$ by Lemma 8.1 for all $m \geq 14$. That is $m_1 \leq 24$. \square

We now study the birationality of pluricanonical maps. The general strategy is to find a small m_0 such that $P_{m_0} \geq 2$ or ≥ 3 . Then we study the map φ_{m_0} according to its type by the method in Section 2. We keep the same notation as in Section 2.

We will not study φ_{m_0} of type I_q for the results of the first author and C. Hacon [4] are effective enough. We will not study φ_{m_0} of type I_n either, because we can not improve Theorem 0.1 of the second author [9] (cf. Theorem 8.4) in this situation.

The structures of the proof for various situations are the same. However, we need to pick different linear systems in different situations.

Lemma 8.12. *Let X be a minimal projective 3-fold of general type with $P_{m_0} \geq 2$. Keep the same notation as in 2.6. Then Assumptions 2.9(3) holds whenever $m \geq m_0 + m_1$ unless the induced map is of type I_q .*

Proof. We consider the linear system $|K_{X'} + \lceil t\pi^*(K_X) \rceil + M_{m_0}| \subset |(m_0 + t + 1)K_{X'}|$ for any $t > 0$. Because $K_{X'} + \lceil t\pi^*(K_X) \rceil \geq (t + 1)\pi^*(K_X)$, we see that $K_{X'} + \lceil t\pi^*(K_X) \rceil$ is effective whenever $t + 1 \geq m_1$.

Now Remark 2.3 says $|K_{X'} + \lceil t\pi^*(K_X) \rceil + M_{m_0}|$ can separate different generic irreducible elements S except when $\dim(B) = 1$ and $b = g(B) > 0$. \square

Proposition 8.13. *Let X be a minimal projective 3-fold of general type with $P_{m_0} \geq 2$. Suppose that the induced map from φ_{m_0} is of type III. Then φ_m is birational for all $m \geq 3m_0 + 1$.*

Proof. Pick a general member S . Take $G := S|_S$. Then $|G|$ is base point free and $\Phi_{|G|}$ gives a generically finite map. So a general member $C \in |G|$ is a smooth curve. Recall that we have $p = 1$. Because $m_0\pi^*(K_X)|_S \geq S|_S \sim C$, we can take $\beta = \frac{1}{m_0}$. So far, by Proposition 3.5(i) and Lemma 8.12(i), Assumptions 2.9(1), (2) and (3) are satisfied for all $m \geq 3m_0$.

We verify Assumptions 2.9(4) under the condition $m \geq 3m_0$. Because

$$\begin{aligned} & K_S + \lceil (m - 1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0} \rceil|_S \\ & \geq K_S + (m - 1)\pi^*(K_X)|_S - (S + E'_{m_0})|_S \\ & = K_S + (m - m_0 - 1)\pi^*(K_X)|_S \geq (m - m_0)\pi^*(K_X)|_S + C. \end{aligned}$$

and $(m - m_0)\pi^*(K_X)|_S \geq 0$ for $m - m_0 \geq 2m_0$ by Proposition 3.5(i), we see that $|K_S + \lceil (m - 1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0} \rceil|_S|$ separates different C since $|C|$ is not composed with any pencils. So Assumptions 2.9(4) is also satisfied.

Now we begin to verify the numerical conditions for α . Because $|G|$ is not composed with any pencils, we see $C^2 = G^2 \geq 2$. Then $\deg(K_C) = (K_S + C) \cdot C \geq (\pi^*(K_X)|_S + 2C) \cdot C > 4$. The evenness of $\deg(K_C)$ says $\deg(K_C) \geq 6$. If we take a sufficiently big m such that $\alpha > 1$, then Remark 2.13 gives $\xi \geq \frac{6}{2m_0+1}$. Take $m \geq 3m_0 + 1$. Then $\alpha = (m - 2m_0 - 1)\xi \geq \frac{6m_0}{2m_0+1} > 2$. Theorem 2.12(II) says that φ_m is birational for all $m \geq 3m_0 + 1$. \square

Proposition 8.14. *Let X be a minimal projective 3-fold of general type with $P_{m_0} \geq 2$. Suppose that the induced map from φ_{m_0} is of type II. Then φ_m is birational for all $m \geq 4m_0 + 2$.*

Proof. Pick a general member S . Take $G := S|_S$. Then $|G|$ is base point free and $|G|$ is composed with a pencil of curves, namely $G^2 = 0$. A generic irreducible element C of $|G|$ is a smooth curve of genus ≥ 2 . Recall that we have $p = 1$. For the same reason we can take $\beta = \frac{1}{m_0}$ since $m_0\pi^*(K_X)|_S \geq G$. So far, by Proposition 3.5(ii) and Lemma 8.12(ii), Assumptions 2.9(1), (2) and (3) are satisfied for all $m \geq 3m_0$.

We verify Assumptions 2.9(4). As we have seen in (i), there are the relations:

$$\begin{aligned} & K_S + \lceil (m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0} \rceil|_S \\ & \geq \pi^*(K_X)|_S + G + (m-m_0-1)\pi^*(K_X)|_S \\ & = (m-m_0)\pi^*(K_X)|_S + G. \end{aligned}$$

Whenever $|G|$ is composed with a rational pencil and $m-m_0 \geq m_1$, e.g. $m \geq 3m_0$, the linear system $|K_S + \lceil (m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0} \rceil|_S|$ can separate different C . Whenever $|G|$ is an irrational pencil, we know that $G \equiv rC$ for $r \geq 2$. We pick two different generic irreducible elements C_1 and C_2 in $|G|$. Since $m_0\pi^*(K_X)|_S - E'_{m_0}|_S \sim G \equiv rC$, we see $\frac{2m_0}{r}\pi^*(K_X)|_S \equiv \frac{2}{r}E'_{m_0}|_S + C_1 + C_2$. Therefore, if we set $Q_m := ((m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0})|_S$, we see that

$$Q_m - \frac{2}{r}E'_{m_0}|_S - C_1 - C_2 \equiv (m-m_0 - \frac{2m_0}{r} - 1)\pi^*(K_X)|_S$$

is nef and big whenever $m \geq 2m_0 + 2$. So the Kawamata-Viehweg vanishing theorem gives $H^1(S, K_S + \lceil Q_m - \frac{2}{r}E'_{m_0}|_S \rceil - C_1 - C_2) = 0$ and thus the surjective map:

$$\begin{aligned} & H^0(S, K_S + \lceil Q_m - \frac{2}{r}E'_{m_0}|_S \rceil) \\ & \longrightarrow H^0(C_1, K_{C_1} + D_1) \oplus H^0(C_2, K_{C_2} + D_2), \end{aligned}$$

where $D_i := \lceil Q_m - \frac{2}{r}E'_{m_0}|_S - C_1 - C_2 \rceil|_{C_i}$ for $i = 1, 2$ with

$$\begin{aligned} \deg(D_i) & \geq (Q_m - \frac{2}{r}E'_{m_0}|_S - C_1 - C_2) \cdot C_i \\ & = (m-m_0 - \frac{2m_0}{r} - 1)\pi^*(K_X)|_S \cdot C_i > 0. \end{aligned}$$

The Riemann-Roch on C_i says $h^0(C_i, K_{C_i} + D_i) > 0$. We thus see that $|K_S + \lceil Q_m - \frac{2}{r}E'_{m_0}|_S \rceil|$ can separate C_1 and C_2 . To be a bigger linear system, $|K_S + \lceil (m-1)\pi^*(K_X) - S - \frac{1}{p}E'_{m_0} \rceil|_S|$ can also separate C_1 and C_2 .

In a word, we have seen that Assumptions 2.9(1), (2), (3) and (4) are all satisfied for $m \geq \max\{3m_0, 2m_0 + 2\}$.

We begin to verify numerical conditions for α . Now if we take $m \gg 0$ such that $\alpha > 1$, Remark 2.13 gives $\xi \geq \frac{2}{2m_0+1}$. Take $m \geq 4m_0 + 3$. Then $\alpha = (m - 2m_0 - 1)\xi > 2$. Theorem 2.12(I) says $\xi \geq \frac{5}{4m_0+3}$. Take $m = 4m_0 + 2$. Then $\alpha = \frac{10m_0+5}{4m_0+3} > 2$. So φ_{4m_0+2} is birational by Theorem 2.12(II). \square

Proposition 8.15. *Let X be a minimal projective 3-fold of general type with $P_{m_0} \geq 2$. Suppose that the induced map from φ_{m_0} is of type I_p . Then φ_m is birational for all $m \geq 4m_0 + 5$.*

Proof. We have an induced fibration $f : X' \rightarrow B$ with $g(B) = 0$. By assumption, $p_g(S) > 0$ for a general fiber S of f . An established theorem (see Bombieri [2], Reid [28], Catanese-Ciliberto [3], and P. Francia [14] or directly refer to Theorem 3.1 in the survey article by Ciliberto [13]), $|2K_{S_0}|$ is always base point free whenever $p_g(S) > 0$. Thus $|2\sigma^*(K_{S_0})|$ is also base point free. We take $G := 2\sigma^*(K_{S_0})$. A generic irreducible element C is a smooth complete curve. On the other hand we have already known there is a sequence of rational numbers $\{\beta_n\}$ with $\beta_n \mapsto \frac{p}{m_0+p}$ such that $\pi^*(K_X)|_S \geq \beta_n \sigma^*(K_{S_0}) = \frac{\beta_n}{2}C$. We can take a $\beta \mapsto \frac{p}{2m_0+2p} \geq \frac{1}{2m_0+2}$ where we know $p = a_{m_0} \geq 1$. So far, by Proposition 3.5(iii) and Lemma 8.12(iii), Assumptions 2.9(1), (2) and (3) are satisfied for all $m \geq m_0 + m_1$. In particular, $m \geq 3m_0 + 3$ will do.

We begin to verify Assumptions 2.9(4). Note that

$$\begin{aligned} & (m - m_0 - 1)\pi^*(K_X)|_S - (m - m_0 - 1)H_n - 2\sigma^*(K_{S_0}) \\ \equiv & \sigma^*(K_{S_0}) + ((m - m_0 - 1)\beta_n - 3)\sigma^*(K_{S_0}). \end{aligned}$$

When $m \geq 4m_0 + 5$ and take a very big n , $(m - m_0 - 1)\beta_n - 3 > 0$. Lemma 3.4(i) says

$$h^0(K_S + \lceil (m - m_0 - 1)\pi^*(K_X)|_S - (m - m_0 - 1)H_n \rceil - 2\sigma^*(K_{S_0})) > 0.$$

Therefore

$$K_S + \lceil (m - m_0 - 1)\pi^*(K_X)|_S \rceil \geq K_S + \lceil (m - m_0 - 1)\pi^*(K_X)|_S \rceil \geq C.$$

So Assumptions 2.9(4) is satisfied for $m \geq 4m_0 + 5$. Simultaneously Assumptions 2.9(1), (2) and (3) are also satisfied.

Now let us verify the numerical conditions for α . We have $\deg(K_C) = (K_S + C) \cdot C > 4K_{S_0}^2 \geq 4$. Because it is even, $\deg(K_C) \geq 6$. We see $\xi = \pi^*(K_X)|_S \cdot C \geq 2\beta_n K_{S_0}^2 \geq 2\beta_n$. Taking limits we have $\xi \geq \frac{2}{m_0+1}$.

Take $m \geq 4m_0 + 5$. Then $\alpha \geq (m - 3m_0 - 3)\xi \geq \frac{2m_0+4}{m_0+1} > 2$. So Theorem 2.12(II) says that φ_m is birational for all $m \geq 4m_0 + 5$. \square

We need the following lemma to prove another result.

Lemma 8.16. *Let S be a nonsingular projective surface of general type. Denote by $\sigma : S \rightarrow S_0$ the blow down onto the minimal model S_0 . Let $|G|$ be the movable part of $|2K_S|$ and C a generic irreducible element of $|G|$. Assume that $|G|$ is base point free. Then $\sigma^*(K_{S_0}) \cdot C \geq 2$.*

Proof. We say that S is of $(1,0)$ type if $K_{S_0}^2 = 1$ and $p_g(S) = 0$. When $p_g(S) > 0$, then $|2\sigma^*(K_{S_0})|$ is base point free as stated in the proof of Proposition 8.15. So $G = 2\sigma^*(K_{S_0})$ and $\sigma^*(K_{S_0}) \cdot C \geq 2$ follows. Thus we only have to study a surface S with $P_g(S) = 0$.

(1) First we study the $(1,0)$ type surface S . Because $P_2(S) = K_{S_0}^2 + \chi(\mathcal{O}_{S_0}) = 2$, we see $h^0(S, G) = 2$. On the other hand a $(1,0)$ type surface S has $q(S) = 0$ (see [2]). Thus $|G|$ is a rational pencil and $C \sim G$. Set $\overline{C} = \sigma_*(C)$. Clearly $h^0(S_0, \overline{C}) \geq h^0(S, C)$. Thus \overline{C} moves in a family. Because $|C|$ is the movable part of $|2K_S|$, $|\overline{C}|$ must be the movable part of $|2K_{S_0}|$ since $P_2(S) = P_2(S_0) \geq h^0(S_0, \overline{C})$. We can write $2K_{S_0} \sim \overline{C} + \overline{Z}_2$ where \overline{Z}_2 is the fixed part.

If $\overline{C}^2 = 0$, then $|\overline{C}|$ is base point free and \overline{C} must be smooth and $\sigma^*(K_{S_0}) \cdot C = K_{S_0} \cdot \overline{C} \geq 2g(\overline{C}) - 2 \geq 2$, noting that \overline{C} is movable in a family which means $g(\overline{C}) \geq 2$.

If $\overline{C}^2 > 0$ and $K_{S_0} \cdot \overline{C} = 1$, then $\overline{C}^2 \leq \frac{(K_{S_0} \cdot \overline{C})^2}{K_{S_0}^2} = 1$. Clearly $\overline{C}^2 = 1$ implies that \overline{C} is smooth. This already says $G(C) = g(\overline{C}) = 2$. The Hodge index theorem says $\overline{C} \equiv K_{S_0}$. So $Z_2 \equiv K_{S_0}$. According to Bombieri [2] or [1], $|3K_{S_0}|$ gives a birational map. So $\Phi_{|3K_{S_0}|}|_{\overline{C}}$ is birational for a general \overline{C} . Because $Z_2 \equiv K_{S_0}$ is nef and big, one has $H^1(S_0, K_{S_0} + Z_2) = 0$ by the Kodaira vanishing. So there is the following surjective map:

$$H^0(S_0, 3K_{S_0}) \rightarrow H^0(\overline{C}, K_{\overline{C}} + Z_2|_{\overline{C}}).$$

Since Z_2 is effective and $Z_2 \cdot \overline{C} = K_{S_0}^2 = 1$, $Z_2|_{\overline{C}}$ is a single point. So the Riemann-Roch on \overline{C} gives $h^0(K_{\overline{C}} + Z_2|_{\overline{C}}) = 2$. Thus the linear system $|K_{\overline{C}} + Z_2|_{\overline{C}}|$ can only give a finite map onto \mathbb{P}^1 , a contradiction. Therefore $\sigma^*(K_{S_0}) \cdot C = K_{S_0} \cdot \overline{C} > 1$.

(2) Assume S is not of $(1,0)$ type. Then $K_{S_0}^2 \geq 2$. We still keep the same notation as in (1). If $\overline{C}^2 = 0$, then \overline{C} must be smooth and $\sigma^*(K_{S_0}) \cdot C = K_{S_0} \cdot \overline{C} \geq 2g(\overline{C}) - 2 \geq 2$.

If $\overline{C}^2 > 0$, then Hodge index theorem says

$$\sigma^*(K_{S_0}) \cdot C = K_{S_0} \cdot \overline{C} \geq \sqrt{K_{S_0}^2} > 1.$$

We are done. \square

Proposition 8.17. *Let X be a minimal projective 3-fold of general type with $P_{m_0} \geq 3$. Suppose that the induced map from φ_{m_0} is of type I_n or I_p . Then φ_m is birational for all $m \geq 3m_0 + 6$.*

Proof. We still take G to be the movable part of $|2K_{S_0}|$. A different point from previous propositions is that $|G|$ is not always base point free. But since we have the induced fibration $f : X' \rightarrow B$, we can consider the relative bi-canonical map of f , namely the rational map $\Psi : X' \dashrightarrow \mathbf{P}$ over B . First we can blow up the indeterminacy of Ψ on X' . Then we can assume, in the birational equivalence sense, that Ψ is a morphism over B . By further modifying π , we can even finally **assume** that π dominates Ψ . With this assumption, we see that $|G|$ is base point free since $|G|$ gives the bicanonical morphism for each fiber S of f . Under the assumption $P_{m_0} \geq 3$, we have $p \geq 2$ and we can take a much better β . In fact we take a sequence $\{\beta_n\}$ with $\beta_n \mapsto \frac{p}{m_0+p} \geq \frac{2}{m_0+2}$ which should give us better bounds. So far, by Proposition 3.5(vi) and Lemma 8.12(vi), Assumptions 2.9(1), (2) and (3) are satisfied for all $m \geq m_0 + m_1$. In particular, $m \geq 3m_0 + 4$ will do.

We study Assumptions 2.9(4). Note that

$$\begin{aligned} & (m - m_0 - 1)\pi^*(K_X)|_S - (m - m_0 - 1)H_n - 2\sigma^*(K_{S_0}) \\ \equiv & 2\sigma^*(K_{S_0}) + ((m - m_0 - 1)\beta_n - 4)\sigma^*(K_{S_0}). \end{aligned}$$

When $m \geq 3m_0 + 6$ and take a very big n , $(m - m_0 - 1)\beta_n - 4 > 0$. Similar to the argument in (iv), Lemma 3.4(ii) and Remark 2.3 tells that Assumptions 2.9(4) is satisfied for $m \geq 3m_0 + 6$. Thus Assumptions 2.9(1), (2) and (3) are all satisfied for $m \geq 3m_0 + 6$.

Now we consider α . By lemma 8.16, we know $\sigma^*(K_{S_0}) \cdot C \geq 2$. Thus $\xi \geq \beta_n \sigma^*(K_{S_0}) \cdot C$. Taking limits one sees $\xi \geq \frac{4}{m_0+2}$. Recall that we may take $\beta \mapsto \frac{p}{2m_0+2p} \geq \frac{1}{m_0+2}$.

Take $m \geq 3m_0 + 6$. Then $\alpha > 2$. So Theorem 2.12(II) says that φ_m is birational for all $m \geq 3m_0 + 6$. \square

Example 8.18. In the case of type II, if $m_0 \geq 13$, then one can easily verify that φ_m is birational for $m \geq 4m_0 - 6$.

Theorem 8.19. *Let X be a minimal projective 3-fold of general type. Then φ_m is birational for all $m \geq 77$.*

Proof. We proceed as the following steps.

Step 1. By Theorem 8.5 and [4], it remains to consider the situation $\chi \geq 1$ and $q(X) = 0$.

Step 2. If $P_{m_0} \geq 2$ for some $m_0 \leq 14$, then φ_m is birational for all $m \geq 76$ by Theorem 8.4.

Step 3. If $\chi \geq 2$, there is an $m_0 \leq 18$ with $P_{m_0} \geq 2$ by Theorem 8.10. Notice that the induced fibration can not be of type I_n by Remark 3.6. Thus φ_m is birational for all $m \geq 77$ by Propositions 8.13, 8.14, 8.15.

Step 4. If $\chi = 1$, then by the classification in Section 6, we have $P_{m_0} \geq 2$ for some $m_0 \leq 14$ except the case I-1. Thus it remains to study the case I-1.

Step 5. For the case I-1, computation shows that $P_{20}(X) \geq P_{20}(B_{min}) = 3$. We consider f which is induced from φ_{20} . If f is of type III, then we get birationality for $m \geq 61$ by Proposition 8.13. If f is of type II, then we get birationality for $m \geq 74$ by Example 8.18. If f is of type I_3 , then we get birationality for $m \geq 66$ by Proposition 8.17. This completes the proof. \square

9. Fletcher's conjecture

First let us recall some notations and definitions in [16].

Let a_0, \dots, a_n be positive integers. Define $S = S(a_0, \dots, a_n)$ to be the graded polynomial ring $\mathbb{C}[x_0, \dots, x_n]$, graded by $\deg(x_i) = a_i$ for all i . The weighted projective space $\mathbb{P}(a_0, a_1, \dots, a_n)$ is defined by $\text{Proj}(S)$. We only consider the *well formed* weighted projective space $\mathbb{P}(a_0, \dots, a_n)$, i.e.

$$\text{hcf}(a_0, \dots, \hat{a}_i, \dots, a_n) = 1 \text{ for each } i.$$

It is clear that the usual projective space $\mathbb{P}^n = \text{Proj}(T)$ where $T = \mathbb{C}[y_0, \dots, y_n]$ and the y_i has weight 1 for all i . Consider the inclusion $S \hookrightarrow T$ given by $x_i \mapsto y_i^{a_i}$ for all i . By [16, p108, 5.12], the induced quotient map $q : \mathbb{P}^n \rightarrow \mathbb{P}(a_0, \dots, a_n)$ is a ramified Galois covering with Galois group $\oplus \mathbb{Z}_{a_i}$. If $\{Y_i\}$ are the coordinates on \mathbb{P}^n , the map q is defined as $[Y_0, \dots, Y_n] \mapsto [Y_0^{a_0}, \dots, Y_n^{a_n}]$.

We consider a hypersurface X of degree d in $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$. Then X has only quotient singularities (locally \mathbb{C}^n by a finite group action) and so the dualizing sheaf $\omega_X \cong i_* \omega_{X_0} = \mathcal{O}_X(K_X)$ where $i : X^0 \hookrightarrow X$ is the inclusion from the smooth part X^0 and K_X (a canonical Weil divisor) is a \mathbb{Q} -Cartier divisor. We would like to classify well formed hypersurfaces which is equivalent to say, by [16, p110, 6.10]:

- (i) $\text{hcf}(a_0, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_n) \mid d$;
- (ii) $\text{hcf}(a_0, \dots, \hat{a}_i, \dots, a_n) = 1$, for all distinct i, j .

From now on we assume $\dim(X) = 3$, $n = 4$ and $d - \sum_i a_i = 1$, which are the assumptions of Fletcher. By 6.14, p111 in [16], one has $\omega_X \cong \mathcal{O}_X(d - \sum a_i) = \mathcal{O}_X(1)$.

If X has only cyclic terminal quotient singularities, then $K_X^3 = \mathcal{O}_X(1)^3$ makes sense. We have $q^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^n}(1)$. Thus $\mathcal{O}_{\mathbb{P}^n}(1)^4 = \frac{1}{a_0 a_1 a_2 a_3 a_4}$. This simply gives

$$K_X^3 = \mathcal{O}_X(1)^3 = d \cdot \mathcal{O}_{\mathbb{P}^n}(1)^4 = \frac{d}{a_0 a_1 a_2 a_3 a_4}. \quad (9.1)$$

Let A be the homogeneous coordinate ring of X . By Lemma 7.1 of [16],

$$P_m(X) = h^0(X, \mathcal{O}_X(m)) = \dim_k A_m. \quad (9.2)$$

In particular, when $m < d$, then $P_m(X) = \dim_k S_m$ clearly, where A_m, S_m denotes the m -th graded part of A, S respectively.

We recall Fletcher's criterion:

Theorem 9.1. [16, p145, 14.1] *Let X_d be a general hypersurface of degree d in $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ and let $\alpha = d - \sum a_i$. Then X_d is quasismooth with only isolated terminal quotient singularities and is not a linear cone if and only if all the following holds:*

- (1) For all i ,
 - (i) $d > a_i$.
 - (ii) there exists a monomial $x_i^m x_e$ of degree d (that is, there exists e such that $a_i | d - a_e$).
 - (iii) if $a_i \nmid d$, there exists an $m \neq i, e$ such that $a_i | a_m + \alpha$.
- (2) For all distinct i, j , with $h = \text{hcf}(a_i, a_j)$, then
 - (i) $h | d$.
 - (ii) there exists an $m \neq i, j$ such that $h | a_m + \alpha$.
 - (iii) one of the following holds:
 - either there exists a monomial $x_i^m x_j^n$ of degree d , or there exist monomials $x_i^{n_1} x_j^{m_1} x_{e_1}$ and $x_i^{n_2} x_j^{m_2} x_{e_2}$ of degree d such that e_1, e_2 are distinct.
 - (iv) there exists a monomial of degree d which does not involve x_i or x_j .
- (3) For all distinct i, j, k , $\text{hcf}(a_i, a_j, a_k) = 1$.

Fletcher has given a list of 23 families (See [16, p150, 15.1]) of quasismooth 3-fold hypersurfaces with only terminal quotient singularities with $\omega_X \cong \mathcal{O}_X(1)$ and $\sum a_i \leq 100$. We would like to prove that Fletcher's list is complete without constraint to the $\sum a_i$.

The main idea of the proof is that, when d is big, the canonical volume of X tends to be very small. But on the other hand we have some effective lower bounds for the volume as proved in Section 3. This helps us to exclude any possibility.

Proof of Theorem 1.5. Consider the canonical hyper-surface 3-fold $X \subset \mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ with $0 < a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4$. Assume $d = \deg(X) = a_0 + a_1 + a_2 + a_3 + a_4 + 1$. Clearly one sees $d \geq 5a_0 + 1$. We have already seen the equality: $K_X^3 = \frac{d}{\pi}$ where $\pi := a_0 a_1 a_2 a_3 a_4$. Explicitly one has:

$$K^3 = \frac{1}{\pi} + \frac{1}{a_1 a_2 a_3 a_4} + \frac{1}{a_0 a_2 a_3 a_4} + \frac{1}{a_0 a_1 a_3 a_4} + \frac{1}{a_0 a_1 a_2 a_4} + \frac{1}{a_0 a_1 a_2 a_3}.$$

Thus if we have $a_i \geq c_i > 0$ for $i = 0, 1, 2, 3, 4$, then we get the inequality:

$$K^3 \leq \frac{c_0 + c_1 + c_2 + c_3 + c_4 + 1}{c_0 c_1 c_2 c_3 c_4}. \quad (9.3)$$

We will frequently use this inequality in the following discussion.

Furthermore since $d \leq 5a_4 + 1$, one gets

$$a_4 \geq \frac{1}{5}(d - 1) \geq 20.$$

We only have to consider the case $d \geq 101$ according to Fletcher's work (see 15.1, p150 in [16]). And we will show that there is no such variety with $d \geq 101$. This verifies the conjecture.

Claim 1. We may assume that $a_0 \leq 4$.

Suppose that $a_0 \geq 5$. Then by Theorem 9.1.3, we have either $a_1 \geq 5, a_2 \geq 6, a_3 \geq 7$ or $(a_0, a_1, a_2, a_3) = (5, 5, 6, 6)$.

If $(a_0, a_1, a_2, a_3) = (5, 5, 6, 6)$, then $a_4 \geq 78$. By (9.3), we have $K^3 \leq \frac{5+5+6+6+78+1}{5 \cdot 5 \cdot 6 \cdot 6 \cdot 78} < \frac{1}{420}$. By Theorem 6.12, this is a contradiction.

If $a_1 \geq 5, a_2 \geq 6, a_3 \geq 7$, then since $a_4 \geq \frac{1}{5}(d-1) \geq 20$, it follows that $K^3 \leq \frac{5+5+6+7+20+1}{5 \cdot 5 \cdot 6 \cdot 7 \cdot 20} < \frac{1}{477}$ by (9.3), this is a contradiction as well. This proves the claim. \square

Claim 2. We may assume that $a_0 + a_1 \leq 11$.

If $a_0 = 1$ and $a_1 \geq 11$, then $a_2 \geq 11, a_3 \geq 12$. We have $K^3 \leq \frac{56}{1 \cdot 11 \cdot 11 \cdot 12 \cdot 20} < \frac{1}{420}$.

If $a_0 = 2$ and $a_1 \geq 8$, then $a_2 \geq 9, a_3 \geq 9$. We have $K^3 \leq \frac{49}{2 \cdot 8 \cdot 9 \cdot 9 \cdot 20} < \frac{1}{420}$.

If $a_0 = 3$ and $a_1 \geq 7$, then $a_2 \geq 7, a_3 \geq 8$. We have $K^3 \leq \frac{46}{3 \cdot 7 \cdot 7 \cdot 8 \cdot 20} < \frac{1}{420}$.

If $a_0 = 4$ and $a_1 \geq 6$, then $a_2 \geq 7, a_3 \geq 7$. We have $K^3 \leq \frac{45}{4 \cdot 6 \cdot 7 \cdot 7 \cdot 20} < \frac{1}{420}$. All of these can not happen by Theorem 6.12. This proves the claim. \square

Because a_4 is the biggest one among a_i , Theorem 9.1.1.(iii) yields that either $a_4 = a_m + 1$ or $a_4 | d$. In the second case, if $d = a_0 + a_1 + a_2 + a_3 + a_4 + 1 = na_4$ for an integer $n > 1$ then it's clear that $n \leq 5$.

Case 1. $d = 5a_4$, then $a_4 \geq 21$ and $\sum_{i=0}^3 (a_4 - a_i) = 1$. It follows that $a_0 \geq a_4 - 1 \geq 20$. Which is absurd.

Case 2. $d = 4a_4$, then $a_4 \geq 26$. Since $a_0 \leq 4$, we have $\sum_{i=1}^3 (a_4 - a_i) \leq 5$. Hence $a_1 \geq a_4 - 5 \geq 21$, which is absurd.

Case 3. $d = 3a_4$, then $a_4 \geq 34$. Since $a_0 + a_1 \leq 11$, we have $(a_4 - a_3) + (a_4 - a_2) \leq 12$. Thus $a_2 \geq a_4 - 12 \geq 22$.

If $a_0 \geq 2$, then $a_1 \geq 2$, we get $K^3 \leq \frac{2+2+22+22+34+1}{4 \cdot 22^2 \cdot 34} < \frac{1}{420}$, a contradiction.

If $a_0 = 1$ and $a_1 \geq 3$, then similarly we have $K^3 \leq \frac{1+3+44+34+1}{1 \cdot 3 \cdot 22^2 \cdot 34} < \frac{1}{420}$, a contradiction.

So now consider the case that $(a_0, a_1) = (1, 1)$ or $(1, 2)$. Notice that $P_2(X) \geq 2$ by (9.2). Hence by Theorem 3.1.iii, we have $K^3 \geq \frac{5}{96}$. On the other hand, by (9.3), we get $K^3 \leq \frac{1+2+44+34+1}{1 \cdot 2 \cdot 22^2 \cdot 34} < \frac{5}{96}$, or $K^3 \leq \frac{1+1+44+34+1}{1 \cdot 1 \cdot 22^2 \cdot 34} < \frac{5}{96}$. Hence both cases lead to a contradiction.

Case 4. We consider the case $n = 2$. Then we have the relation: $a_4 \geq 51$ and $a_0 + a_1 + a_2 + a_3 + 1 = a_4 \geq 51$. Since $a_0 + a_1 \leq 11$ by Claim 2, we have $2a_3 \geq a_2 + a_3 \geq 39$ and thus $a_3 \geq 20$.

If $a_0 = 4$, then we must have $a_1 \geq 4$ and $a_2 \geq 5$. Thus we get $K^3 \leq \frac{8+5+20+51+1}{4^2 \cdot 5 \cdot 20 \cdot 51} < \frac{1}{420}$, a contradiction.

If $a_0 = 3$, then we must have $a_1 \geq 3, a_2 \geq 4$. Thus we get $K^3 \leq \frac{6+4+20+51+1}{3^2 \cdot 4 \cdot 20 \cdot 51} < \frac{1}{420}$, a contradiction.

If $a_0 = 2$ and $a_1 \geq 4$, then $a_2 \geq 5$ and we have $K^3 \leq \frac{2+4+6+20+51+1}{2 \cdot 4 \cdot 5 \cdot 20 \cdot 51} < \frac{1}{420}$, a contradiction. If $a_1 = 3$ and $a_2 \geq 5$, then we get $K^3 \leq \frac{2+3+5+20+51+1}{2 \cdot 3 \cdot 5 \cdot 20 \cdot 51} < \frac{1}{320}$. On the other hand, since $P_6(X) \geq 2$, Theorem 3.1.iii gives $K^3 \geq \frac{1}{311}$, a contradiction; If $a_1 = 3$ and $a_2 = 4$, then $a_3 \geq 41$. We get $K^3 \leq \frac{1}{492}$, a contradiction; If $a_1 = 3$ and $a_2 = 3$, then $a_3 \geq 42$. We get $K^3 \leq \frac{1}{378}$. Since $P_3(X) \geq 2$, Theorem 3.1 (iii) says $K^3 > \frac{1}{53}$, a contradiction. If $a_1 = 2$, then we can get $K^3 \leq \frac{2+2+2+20+51+1}{2 \cdot 2 \cdot 3 \cdot 20 \cdot 51} < \frac{1}{120}$. Since $P_2(X) \geq 2$, Theorem 3.1.(iii) gives $K^3 > \frac{1}{20}$, a contradiction.

Finally we consider the case that $a_0 = 1$. If $a_1 \geq 6$, then $a_2 \geq 6$. Thus we can get $K^3 \leq \frac{1+6+6+20+51+1}{1 \cdot 6 \cdot 6 \cdot 20 \cdot 51} < \frac{1}{420}$, a contradiction. We thus consider the case that $a_1 \leq 6$. Notice that on one hand, we have $P_{a_1}(X) \geq 2$, hence $K^3 \geq \frac{11}{12a_1(a_1+1)^2}$ by Theorem 3.1.(iii). On the other hand, $K^3 \leq \frac{72+2a_1}{1 \cdot a_1 \cdot a_1 \cdot 20 \cdot 51}$. However, it's easy to verify that for $a_1 \leq 5$, $\frac{11}{12a_1(a_1+1)^2} > \frac{72+2a_1}{1020a_1^2}$. This leads to a contradiction.

It thus remain to consider the case that $a_4 = a_m + 1$ for some m .

Case 5. If $a_4 = a_3 + 1$, and $a_3 | d$. Say $d = na_3$. Note that $d = a_0 + a_1 + a_2 + 2a_3 + 2 = na_3 \geq 101$. If $n \geq 4$, then one get a contradiction easily for $a_0 + a_1 \leq 11 < a_3 - 2$. If $n = 3$, then $a_2 \geq a_3 - 12 \geq 22$ since $a_3 \geq 34$. Thus one can easily check that $K^3 < \frac{1}{420}$ unless $(a_0, a_1) = (1, 1), (1, 2)$. In both case, one has $K^3 < \frac{5}{96}$. However $P_2 \geq 2$ gives the required contradiction.

Case 6. If $a_4 = a_3 + 1$ and $a_3 = a_m + 1$, then $a_3 = a_2 + 1$ since $a_1 \leq 11$ by Claim 2. Now $a_0 + a_1 + 3a_2 + 3 + 1 \geq 101$ gives $a_2 \geq 29$. If $a_1 \geq 2$, then one can easily check that $K^3 \leq \frac{1+2+29+30+31+1}{1 \cdot 2 \cdot 29 \cdot 30 \cdot 31} < \frac{1}{420}$.

Hence we only need to consider $a_1 = a_0 = 1$. It's easy to see that $K^3 < \frac{1}{3}$. Now $p_g \geq 2$, thus $K^3 \geq \frac{1}{3}$ by [8], this is the required contradiction.

Case 7. Finally, if $a_4 = a_m + 1$ for some $m \leq 2$, then clearly $a_4 = a_2 + 1$ and it follows that $a_3 = a_2$ or $a_3 = a_2 + 1$. The similar argument as in Case 6 gives a contradiction.

This completes the proof. \square

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